Section 2.2. The Limit of a Function

> *restart*

> *Digits* := 15

$$Digits := 15 \quad (1)$$

>

Define the function $f(x)$ as follows:

> $f := x \rightarrow \begin{cases} \dfrac{x-1}{x^2-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

$$f := x \rightarrow piecewise\left(x \neq 1, \frac{x-1}{x^2-1}, x = 1, 2\right) \quad (2)$$

> $f(x)$

$$\begin{cases} \dfrac{-1+x}{x^2-1} & x \neq 1 \\ 2 & x = 1 \end{cases} \quad (3)$$

> $f(-2); f(0); f(2)$

$$-1$$
$$1$$
$$\frac{1}{3} \quad (4)$$

> $f(1)$

$$2 \quad (5)$$

>

How does the function f behave near $x = 1$?

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> $f(0.9); f(0.99), f(0.999), f(0.9999999)$

0.526315789473684

0.502512562814070, 0.500250125062531, 0.500000025000001 **(6)**

> $f(1.1); f(1.01), f(1.001), f(1.0000001)$

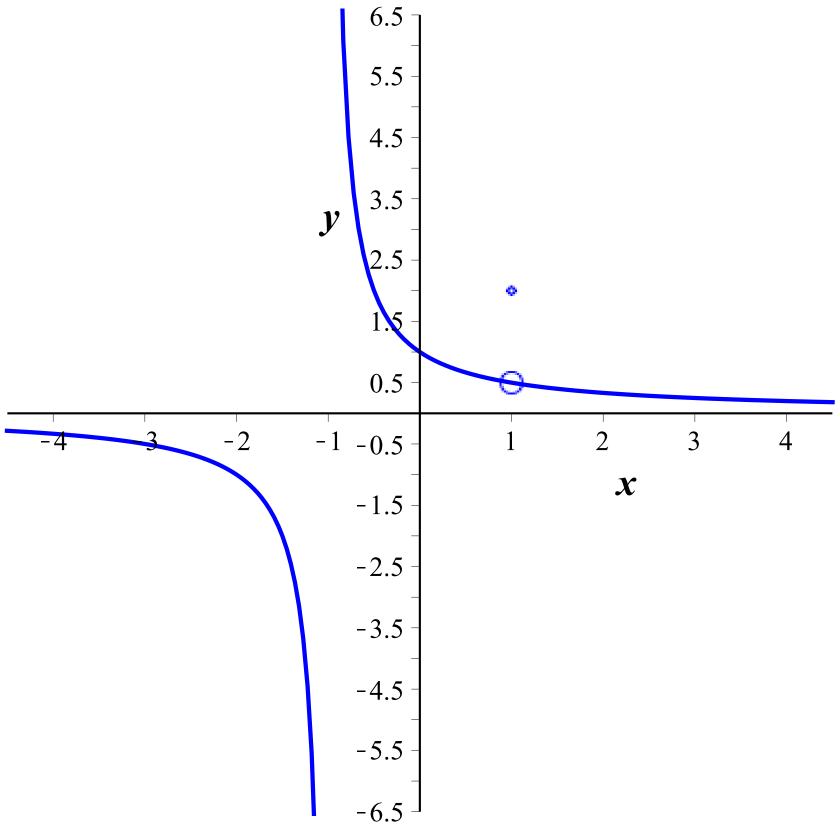
0.476190476190476

0.497512437810945, 0.499750124937531, 0.499999975000001 **(7)**

>

Consider the graph of f:

> *plot*($f(x)$, *view* = [−4.5 ..4.5, −6.5 ..6.5], *labels* = [*x*, *y*], *labelfont* = [*times*, *bold*, 14], *thickness* = 2, *color* = *blue*, *discont* = [*showremovable*], *tickmarks* = [8, 25])



>
Observations:
As $x \to 1^-, f(x) \to 0.5$ (left-hand limit). Alternate notation: $\lim_{x \to 1^-} f(x) = 0.5$.
As $x \to 1^+, f(x) \to 0.5$ (right-hand limit). Alternate notation: $\lim_{x \to 1+} f(x) = 0.5$.

>
Thus, the limit (i.e., the two-sided limit) exists. We write: As $x \to 1, f(x) \to 0.5$.
Or, we write $\lim_{x \to 1} f(x) = 0.5$.

>
NOTE: $f(1) = 2$. Thus, for this function, we have $\lim_{x \to 1} f(x) \neq f(1)$. Because of this, we say that f is **discontinuous** at $x = 1$.

>
Observation: Since $x^2 - 1 = (x - 1)(x + 1)$, the function f can be simplified as follows:

> $f := x \to \begin{cases} \dfrac{1}{x+1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

$$f := x \to piecewise\left(x \neq 1, \frac{1}{x+1}, x = 1, 2\right) \quad \textbf{(8)}$$

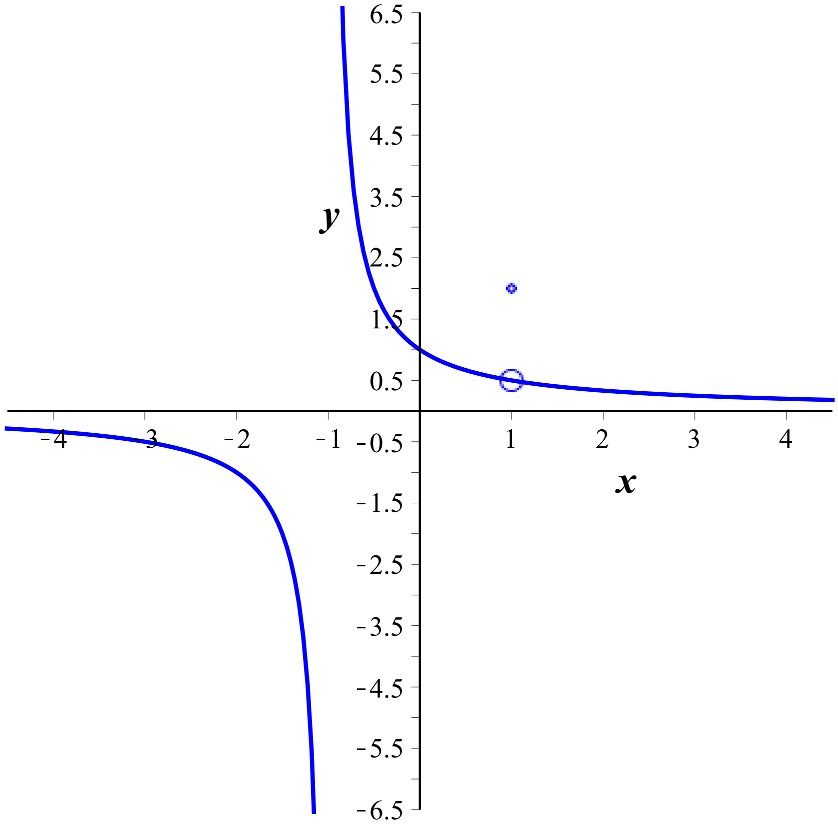
> $f(x)$

$$\begin{cases} \dfrac{1}{x+1} & x \neq 1 \\ 2 & x = 1 \end{cases} \quad \textbf{(9)}$$

>

Using this simplified form, the graph of *f* once again is

> $plot(f(x), view = [-4.5..4.5, -6.5..6.5], labels = [x, y], labelfont = [times, bold, 14], thickness = 2, color = blue, discont = [showremovable], tickmarks = [8, 25])$



>

>

Now consider functions closely related to f, namely g and h, defined as follows:

> $g := x \rightarrow \frac{1}{x}$

$$g := x \rightarrow \frac{1}{x} \qquad \textbf{(10)}$$

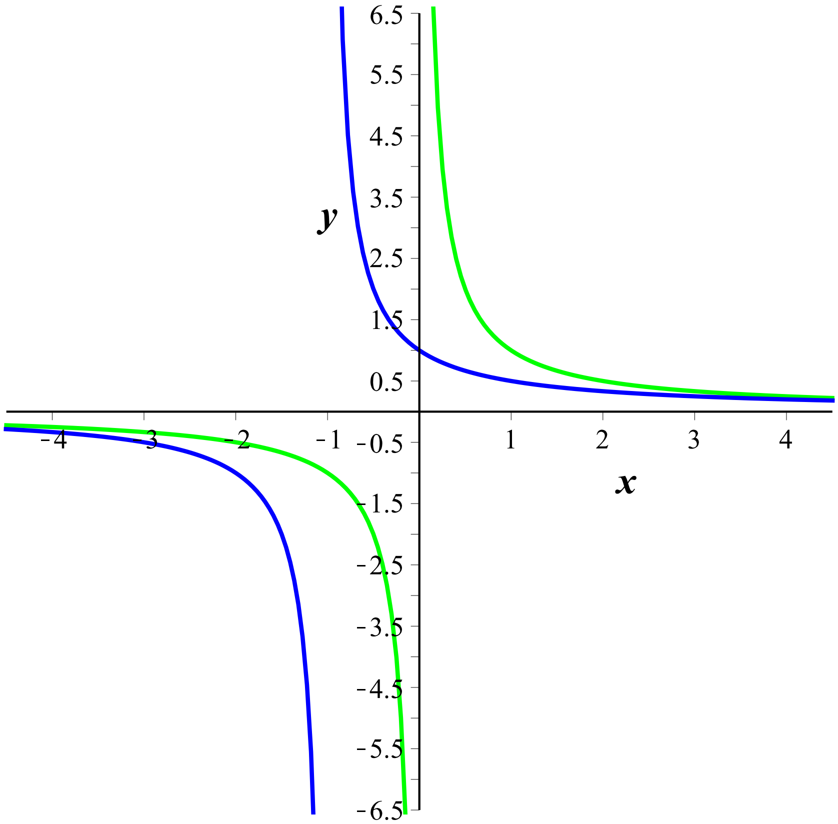
> $h := x \rightarrow \frac{1}{x+1}$

$$h := x \rightarrow \frac{1}{x+1} \qquad \textbf{(11)}$$

>

> *plot*([$g(x)$, $h(x)$], *view* = [−4.5 ..4.5, −6.5 ..6.5], *labels* = [*x*, *y*], *labelfont* = [*times*, *bold*,

14], *thickness* = 2, *color* = [*green*, *blue*], *discont* = [*showremovable*], *tickmarks* = [8, 25])



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Questions:

1. How are g and h related graphically?

2. What is $\lim_{x \to 1^-} g(x)$? (look at the green graph). What is $\lim_{x \to 1+} g(x)$?

3. What is $\lim_{x \to 1} g(x)$?

>

>

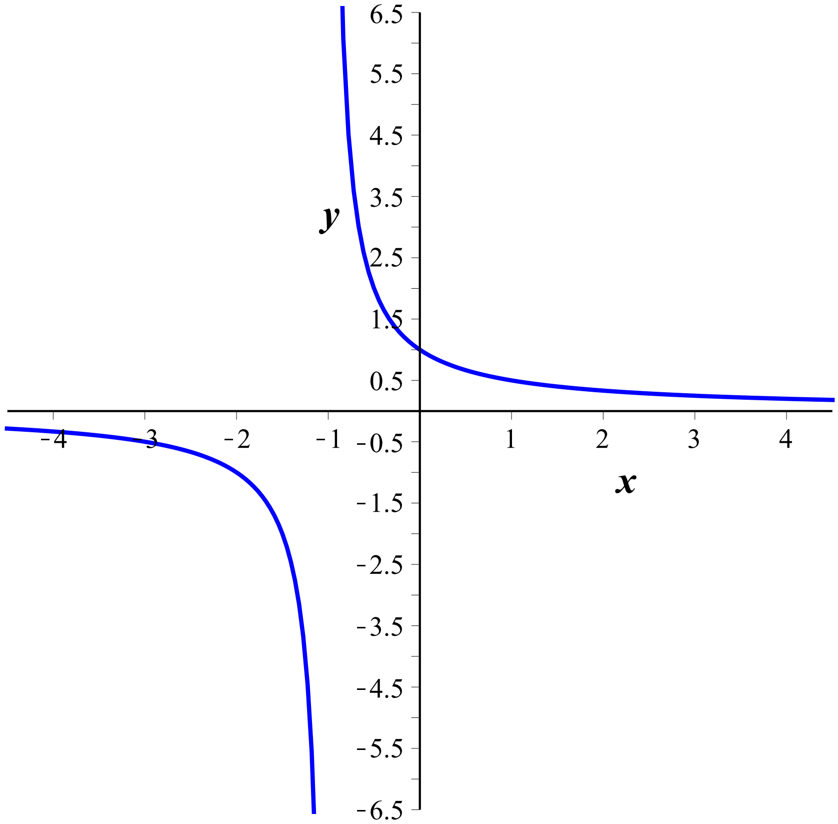
Infinite Limits

How does the function $h(x) = \frac{1}{x+1}$ behave near $x = -1$?

> $h := x \rightarrow \frac{1}{x+1}$

$$h := x \rightarrow \frac{1}{x+1} \quad (12)$$

> plot(h(x), view = [-4.5 ..4.5, -6.5 ..6.5], labels = [x, y], labelfont = [times, bold, 14], thickness = 2, color = [blue], discont = [showremovable], tickmarks = [8, 25])



>

> h(-1)

Error, (in h) numeric exception: division by zero

>

> h(-1.1), h(-1.01), h(-1.001), h(-1.000123)

-10., -100., -1000., -8130.08130081301 **(13)**

> h(-0.9), h(-0.99), h(-0.999), h(-0.9999871)

10., 100., 1000., 77519.3798449612 **(14)**

>

We write $\lim_{x \to -1^+} h(x) = \infty$, which means that the values of $f(x)$ become arbitarily large (or, *increase without bound*) as x approaches -1 from the right. However, as x approaches -1 from the left, then the values of $f(x)$ are negative and $|f(x)|$ becomes arbitarily large. (We say that $f(x)$ *decreases without bound*). To communicate this succiently, we write $\lim_{x \to -1^-} h(x) = -\infty$.

>

*** END ***

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