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10 Exponential and Logarithmic Functions

10–1 Rational Exponents

Objective: To extend the meaning of exponents to include rational numbers.

Vocabulary

Exponential form A radical expression is in exponential form when it is expressed as a power or product of powers.

$b^{1/n}$ The exponential form of the nth root of b. The notation follows from the laws of exponents. Since $(b^{1/n})^n = b^{(1/n)\cdot n} = b^1 = b$, and $(\sqrt[n]{b})^n = b$, we can say that $b^{1/n} = \sqrt[n]{b}$. Examples: $9^{1/2} = \sqrt{9} = 3$ and $-8^{1/3} = -\sqrt[3]{8} = -2$

$b^{p/q}$ The qth root of b to the power p. In other words, if p and q are integers, with $q > 0$, and b is a positive real number, then $b^{p/q} = (\sqrt[q]{b})^p = \sqrt[q]{b^p}$.

CAUTION You will have to be familiar with the laws of exponents from Lesson 5–1 in order to do the exercises in this lesson.

Example 1 Simplify $64^{2/3}$.

Solution 1 $64^{2/3} = (\sqrt[3]{64})^2 = 4^2 = 16$

Solution 2 $64^{2/3} = \sqrt[3]{64^2} = \sqrt[3]{4096} = 16$

Example 2 Simplify.

a. $32^{-3/5}$ **b.** $4^{3.5}$ **c.** $(8^5)^{1/3}$ **d.** $(25^{1/2} + 36^{1/2})^2$

Solution

a. $32^{-3/5} = \dfrac{1}{32^{3/5}} = \dfrac{1}{(\sqrt[5]{32})^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

b. $4^{3.5} = 4^{7/2} = (\sqrt{4})^7 = 2^7 = 128$

c. $(8^5)^{1/3} = 8^{5/3} = (\sqrt[3]{8})^5 = 2^5 = 32$

d. $(25^{1/2} + 36^{1/2})^2 = (\sqrt{25} + \sqrt{36})^2 = (5 + 6)^2 = 11^2 = 121$

Simplify.

1. $16^{1/2}$ **2.** $81^{1/4}$ **3.** $16^{-1/2}$ **4.** $27^{-1/3}$ **5.** $81^{3/4}$ **6.** $36^{-3/2}$

7. $-64^{2/3}$ **8.** $4^{3.5}$ **9.** $\left(\dfrac{9}{16}\right)^{1.5}$ **10.** $25^{-0.5}$ **11.** $4^{-2.5}$ **12.** $(9^{-3})^{1/6}$

13. $(27^5)^{-2/15}$ **14.** $(27^{1/3} + 64^{1/3})^{-2}$ **15.** $(27^{2/3} - 32^{1/5})^2$

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10–1 *Rational Exponents* *(continued)*

Example 3 Write in exponential form: **a.** $\sqrt[6]{x^3y^{-2}}$ **b.** $\sqrt[5]{\dfrac{9a^5}{b^2}}$

Solution **a.** $\sqrt[6]{x^3y^{-2}} = (x^3y^{-2})^{1/6}$

$= x^{3/6}y^{-2/6}$

$= x^{1/2}y^{-1/3}$

b. $\sqrt[5]{\dfrac{9a^5}{b^2}} = \left(\dfrac{9a^5}{b^2}\right)^{1/5}$

$= \dfrac{9^{1/5}a^{5/5}}{b^{2/5}}$

$= 9^{1/5}ab^{-2/5}$

Write in exponential form.

16. $\sqrt{x^5y^4}$ **17.** $\sqrt[5]{x^4y^{-6}}$ **18.** $\sqrt[3]{m^{-3}n^{-6}}$ **19.** $\sqrt[3]{27u^9v^{-6}}$ **20.** $\sqrt[4]{\dfrac{3^4 \cdot a^{-6}}{b^2}}$

Example 4 Express $\sqrt{8} \cdot \sqrt[3]{4}$ in simplest radical form.

Solution

$\sqrt{8} \cdot \sqrt[3]{4} = \sqrt{2^3} \cdot \sqrt[3]{2^2}$ — Express each radical in exponential form.

$= 2^{3/2} \cdot 2^{2/3}$

$= 2^{9/6 + 4/6}$ — Use the rule $a^m \cdot a^n = a^{m+n}$.

$= 2^{13/6}$

$= 2^{12/6} \cdot 2^{1/6}$ — Factor the power into two powers, one with an exponent that is the largest possible whole number.

$= 2^2 \cdot 2^{1/6}$

$= 4\sqrt[6]{2}$ — Write in simplest radical form.

Express in simplest radical form.

21. $\sqrt[3]{25} \cdot \sqrt[3]{25}$ **22.** $\sqrt{27} \cdot \sqrt[6]{27}$ **23.** $\dfrac{\sqrt[3]{9}}{\sqrt[6]{3}}$ **24.** $\dfrac{\sqrt[8]{16}}{\sqrt[6]{4}}$ **25.** $\sqrt[3]{32} \cdot \sqrt{8}$

Mixed Review Exercises

Express in simplest form without negative exponents.

1. $\dfrac{x}{x-2} - \dfrac{8}{x^2-4}$ **2.** $\sqrt{\dfrac{75x}{y^7}}$ **3.** $(a^2 + 2a - 3) - (5 - a^2)$

4. $(-3x^3)(-3x)^3$ **5.** $\left(\dfrac{x^2}{y}\right)^{-3} \cdot \left(\dfrac{y^2}{x^3}\right)^{-1}$ **6.** $(3n + 2)(2n - 5)$

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10–2 Real Number Exponents

Objective: To extend the meaning of exponents to include irrational numbers and to define exponential functions.

Vocabulary

Exponential function If $b > 0$ and $b \neq 1$, the function defined by $y = b^x$ is called the exponential function with base b. Example: $y = 3^x$.

One-to-one function A function f is called one-to-one if for every p and q in the domain of f, $f(p) = f(q)$ if and only if $p = q$.

Exponential equation An equation in which a variable appears in an exponent.

Example 1 Simplify: **a.** $5^{\sqrt{2}} \cdot 5^{\sqrt{2}}$ **b.** $\dfrac{5^{\sqrt{2}+2}}{5^{\sqrt{2}-1}}$ **c.** $\sqrt[3]{2^{9\pi}}$ **d.** $\dfrac{16^{1.4}}{2^{1.6}}$

Solution

a. $5^{\sqrt{2}} \cdot 5^{\sqrt{2}} = 5^{\sqrt{2}+\sqrt{2}}$ Recall that $a^m \cdot a^n = a^{m+n}$.

$= 5^{2\sqrt{2}}$ Simplify the exponent.

$= (5^2)^{\sqrt{2}} = 25^{\sqrt{2}}$ Recall that $(a^m)^n = a^{mn}$.

b. $\dfrac{5^{\sqrt{2}+2}}{5^{\sqrt{2}-1}} = 5^{(\sqrt{2}+2)-(\sqrt{2}-1)}$ Recall that $\dfrac{a^m}{a^n} = a^{m-n}$.

$= 5^3 = 125$ Simplify the exponent.

c. $\sqrt[3]{2^{9\pi}} = 2^{9\pi/3}$ Write in exponential form.

$= 2^{3\pi}$ Simplify the exponent.

$= (2^3)^{\pi} = 8^{\pi}$

d. $\dfrac{16^{1.4}}{2^{1.6}} = \dfrac{(2^4)^{1.4}}{2^{1.6}}$ Write the numerator and denominator using the same base.

$= \dfrac{2^{5.6}}{2^{1.6}}$

$= 2^{5.6-1.6}$

$= 2^4 = 16$ Simplify the exponent.

Simplify.

1. $4^{\sqrt{3}} \cdot 4^{\sqrt{3}}$ **2.** $(4^{\sqrt{3}})^3$ **3.** $(4^{\sqrt{3}})^{\sqrt{3}}$ **4.** $\dfrac{4^{\sqrt{3}+1}}{4^{\sqrt{3}-1}}$

5. $6^{\sqrt{5}} \cdot 6^{\sqrt{3}}$ **6.** $(5^{\sqrt{2}})^2$ **7.** $(9^{\pi})^2$ **8.** $(5^{\sqrt{3}})^{-1/\sqrt{3}}$

9. $\sqrt{5^{4\pi}}$ **10.** $\sqrt[5]{3^{10\pi}}$ **11.** $9^{2.1} \cdot 3^{-3.2}$ **12.** $(\sqrt{7})^{\sqrt{3}}(\sqrt{7})^{-\sqrt{3}}$

13. $\dfrac{7^{\sqrt{5}-3}}{7^{\sqrt{5}-5}}$ **14.** $\dfrac{36^{1.2}}{6^{4.4}}$ **15.** $\sqrt[4]{\dfrac{25^{5+\pi}}{25^{5-\pi}}}$ **16.** $\dfrac{(1+\sqrt{2})^{2+\pi}}{(1+\sqrt{2})^{\pi}}$

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10–2 Real Number Exponents *(continued)*

Vocabulary

Steps for solving exponential equations

1. Express each side of the equation as a power of the same base.
2. Set the exponents equal and then solve.
3. Check the answer.

Example 2 Solve.

a. $27^x = \frac{1}{9}$ **b.** $7^{4-x} = 49^{x-1}$ **c.** $125^{x-3} = 5\sqrt{5}$

Solution

	Step 1	*Step 2*	*Step 3*
a.	$27^x = \frac{1}{9}$	$3x = -2$	$27^{-2/3} \stackrel{?}{=} \frac{1}{9}$
	$(3^3)^x = \frac{1}{3^2}$	$x = -\frac{2}{3}$	$(\sqrt[3]{27})^{-2} \stackrel{?}{=} \frac{1}{9}$
	$3^{3x} = 3^{-2}$		$3^{-2} \stackrel{?}{=} \frac{1}{9}$
	$\therefore$ the solution set is $\left\{-\frac{2}{3}\right\}$.		$\frac{1}{9} = \frac{1}{9}$ $\surd$
b.	$7^{4-x} = 49^{x-1}$	$4 - x = 2x - 2$	$7^{4-2} \stackrel{?}{=} 49^{2-1}$
	$7^{4-x} = (7^2)^{x-1}$	$-3x = -6$	$7^2 \stackrel{?}{=} 49^1$
	$7^{4-x} = 7^{2x-2}$	$x = 2$	$49 = 49$ $\surd$
	$\therefore$ the solution set is $\{2\}$.		
c.	$125^{x-3} = 5\sqrt{5}$	$3x - 9 = \frac{3}{2}$	$125^{(7/2)-3} \stackrel{?}{=} 5\sqrt{5}$
	$(5^3)^{x-3} = 5^1 \cdot 5^{1/2}$	$6x - 18 = 3$	$125^{1/2} \stackrel{?}{=} 5\sqrt{5}$
	$5^{3x-9} = 5^{3/2}$	$6x = 21$	$\sqrt{125} \stackrel{?}{=} 5\sqrt{5}$
		$x = \frac{21}{6} = \frac{7}{2}$	$5\sqrt{5} = 5\sqrt{5}$ $\surd$
	$\therefore$ the solution set is $\left\{\frac{7}{2}\right\}$.		

Solve. If an equation has no solution, say so.

17. $4^x = \frac{1}{16}$ **18.** $3^x = \sqrt{27}$ **19.** $9^{3+x} = 3$

20. $9^{1+x} = 27$ **21.** $64^{x-1} = 4$ **22.** $100^{2x} = 10^{3x+2}$

23. $27^{x-2} = 3\sqrt{3}$ **24.** $7^{2x-1} = 49^{x-3}$ **25.** $4^{-(1-x)} = 16^x$

26. $36^{x+1} = 6^{x-1}$ **27.** $25^{3-x} = 5^{2x+1}$ **28.** $8^{2x+3} = 64^{x-5}$

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10–3 Composition and Inverses of Functions

Objective: To find the composite of two given functions and to find the inverse of a given function.

Vocabulary

Composite of two functions The function whose value at x is $f(g(x))$ is called the composite of the functions f and g. The operation that combines f and g to produce their composite is called *composition*. Note: In general, $f(g(x)) \neq g(f(x))$.

Identity function The function that maps x to itself, that is, $I(x) = x$.

Inverse function The functions f and g are inverse functions if together they act as the identity function. In other words, f and g are inverse functions if

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

The graph of a function and its inverse are mirror images about the line $y = x$.

Horizontal-line test The inverse of a function is a function if and only if every horizontal line intersects the graph of the function in *at most* one point.

Symbols

f^{-1} means the inverse of a function or "f inverse."

$f^{-1}(x)$ means the value of f inverse at x.

CAUTION The superscript $^{-1}$ used in inverse function notation is *not* an exponent.

Example 1 If $f(x) = 2x - 7$ and $g(x) = \sqrt{x} + 3$, find the following.

a. $f(g(16))$ **b.** $g(f(8))$ **c.** $f(g(x))$

Solution

a. First, find $g(16)$: $g(16) = \sqrt{16} + 3 = 4 + 3 = 7$

Substitute 7 for $g(16)$: $f(g(16)) = f(7) = 2(7) - 7 = 14 - 7$

$\therefore f(g(16)) = 7$

b. First, find $f(8)$: $f(8) = 2(8) - 7 = 16 - 7 = 9$

Substitute 9 for $f(8)$: $g(f(8)) = g(9) = \sqrt{9} + 3 = 3 + 3$

$\therefore g(f(8)) = 6$

c. First, find $g(x)$: $g(x) = \sqrt{x} + 3$

Substitute $\sqrt{x} + 3$ for $g(x)$: $f(g(x)) = f(\sqrt{x} + 3) = 2(\sqrt{x} + 3) - 7$

$\therefore f(g(x)) = 2\sqrt{x} - 1$

If $f(x) = 2x + 1$, $g(x) = \sqrt{x} - 1$, and $h(x) = \frac{x + 1}{2}$, find a real-number value or an expression in x for each of the following. If no real value can be found, say so.

1. a. $f(g(4))$	**b.** $f(g(9))$	**c.** $f(g(0))$	**d.** $f(f(x))$
2. a. $g(f(4))$	**b.** $g(f(9))$	**c.** $g(f(0))$	**d.** $g(f(x))$
3. a. $f(h(5))$	**b.** $f(h(2))$	**c.** $h(f(\sqrt{3}))$	**d.** $f(h(x))$
4. a. $h(g(9))$	**b.** $h(g(-3))$	**c.** $g(h(15))$	**d.** $h(g(x))$

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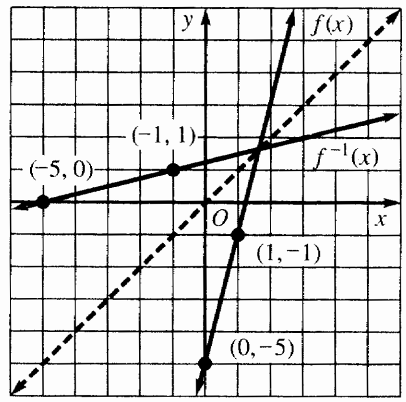
10–3 Composition and Inverses of Functions *(continued)*

Example 2 Graph $f(x) = 4x - 5$ and use the horizontal-line test to determine whether f has an inverse function. If so, graph f^{-1} by reflecting f across the line $y = x$.

Solution The graph of the function $f(x) = 4x - 5$ is a line passing through the points $(0, -5)$ and $(1, -1)$. It passes the horizontal-line test, so f^{-1} is a function.

If a point (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}. Thus, the graph of f^{-1} passes through the points $(-5, 0)$ and $(-1, 1)$.



Example 3 Find f^{-1} for each of the following functions.

a. $f(x) = 2x - 6$ **b.** $f(x) = x^7$

Solution

a. Replace $f(x)$ by y: $y = 2x - 6$

Interchange x and y: $x = 2y - 6$

Solve for y: $x + 6 = 2y$

$y = \frac{1}{2}x + 3$

$\therefore f^{-1}(x) = \frac{1}{2}x + 3$

b. Replace $f(x)$ by y: $y = x^7$

Interchange x and y: $x = y^7$

Solve for y: $\sqrt[7]{x} = \sqrt[7]{y^7}$

$y = \sqrt[7]{x}$

$\therefore f^{-1}(x) = \sqrt[7]{x}$

Graph f and use the horizontal-line test to determine whether f has an inverse function. If so, graph f^{-1} by reflecting f across the line $y = x$, and find f^{-1}. If not, say so.

5. $f(x) = 3x + 2$ **6.** $f(x) = \frac{x + 1}{2}$ **7.** $f(x) = x^2$

8. $f(x) = -x^3$ **9.** $f(x) = \frac{4}{x}$ **10.** $f(x) = |x| + 1$

Mixed Review Exercises

Simplify.

1. $100^{-3/2}$ **2.** $\sqrt{12} + \sqrt{48}$ **3.** $(3 + 2i)^2$

4. $(2^{\sqrt{3}})^{2\sqrt{3}}$ **5.** $\sqrt{-18} \cdot \sqrt{-6}$ **6.** $\frac{\sqrt{3} - 2}{1 + \sqrt{3}}$

7. $\frac{6^2 - 1}{3 - 2^3}$ **8.** $\frac{3 + 2i}{1 - i}$ **9.** $16^{1.5} \cdot 2^{-2}$

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10–4 Definition of Logarithms

Objective: To define logarithmic functions and to learn how they are related to exponential functions.

Vocabulary

Logarithmic function The inverse of an exponential function.
Example: The inverse function of $f(x) = 3^x$ is the logarithmic function $f(x) = \log_3 x$. It is read "the base 3 logarithm of x" or "log base 3 of x."

Logarithm ($\log_b N$) If b and N are positive numbers ($b \neq 1$), then $\log_b N = k$ if and only if $b^k = N$. $\log_b N = k$ is in *logarithmic* form and $b^k = N$ is in *exponential* form.

Properties of logarithms

1. Every positive number has a unique logarithm with base b.
2. $\log_b M = \log_b N$ if and only if $M = N$
3. $\log_b b^k = k$
4. $b^{\log_b N} = N$
5. $\log_b b = 1$
6. $\log_b 1 = 0$

Steps for simplifying logarithms

1. Set the logarithm that you want to simplify equal to k.
2. Write the equation in exponential form.
3. Rewrite each side of the equation using the same base.
4. Set the exponents equal.

Example 1

Exponential Form		*Logarithmic Form*
$3^4 = 81$	means	$\log_3 81 = 4$
$3^2 = 9$	means	$\log_3 9 = 2$
$3^0 = 1$	means	$\log_3 1 = 0$
$3^{-1} = \frac{1}{3}$	means	$\log_3 \frac{1}{3} = -1$
$3^k = N$	means	$\log_3 N = k$

Notice the pattern: $\log_3 81 = 4$ $\quad$ $\log_3 9 = 2$ $\quad$ $\log_3 1 = 0$

Write each equation in exponential form.

1. $\log_2 64 = 6$ $\quad$ **2.** $\log_4 \frac{1}{64} = -3$ $\quad$ **3.** $\log_{10} (0.01) = -2$

Write each equation in logarithmic form.

4. $2^5 = 32$ $\quad$ **5.** $5^{-1/2} = \frac{\sqrt{5}}{5}$ $\quad$ **6.** $10^{-1} = 0.1$

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10–4 Definition of Logarithms *(continued)*

Example 2 Simplify each logarithm.

a. $\log_4 64$ **b.** $\log_3 27\sqrt{3}$ **c.** $\log_2 \frac{1}{8}$

Solution

a. $\log_4 64 = k \longrightarrow 4^k = 64 \longrightarrow 4^k = 4^3 \longrightarrow k = 3$

$\therefore \log_4 64 = 3$

b. $\log_3 27\sqrt{3} = k \longrightarrow 3^k = 27\sqrt{3} \longrightarrow 3^k = 3^{7/2} \longrightarrow k = \frac{7}{2}$

$= 3^3 \cdot 3^{1/2}$

$= 3^{7/2}$

$\therefore \log_3 27\sqrt{3} = \frac{7}{2}$

c. $\log_2 \frac{1}{8} = k \longrightarrow 2^k = \frac{1}{8} \longrightarrow 2^k = 2^{-3} \longrightarrow k = -3$

$= \frac{1}{2^3}$

$= 2^{-3}$

$\therefore \log_2 \frac{1}{8} = -3$

Simplify each logarithm.

7. $\log_2 8$ **8.** $\log_8 64$ **9.** $\log_6 216$ **10.** $\log_7 7$

11. $\log_6 1$ **12.** $\log_8 2$ **13.** $\log_7 \frac{1}{49}$ **14.** $\log_9 \frac{1}{27}$

15. $\log_5 \sqrt{5}$ **16.** $\log_7 7\sqrt{7}$ **17.** $\log_9 \sqrt{3}$ **18.** $\log_8 \sqrt[3]{2}$

19. $\log_{1/2} 16$ **20.** $\log_{1/3} 9$ **21.** $\log_{1/3} 81$ **22.** $\log_3 \sqrt[3]{\frac{1}{9}}$

Example 3 Solve each equation.

a. $\log_5 x = 4$ **b.** $\log_x 32 = 5$

Solution

a. Write in exponential form.

$\log_5 x = 4$

$5^4 = x$

$625 = x$

$\therefore$ the solution set is $\{625\}$.

b. Write in exponential form.

$\log_x 32 = 5$

$x^5 = 32$

$x^5 = 2^5$

$x = 2$

$\therefore$ the solution set is $\{2\}$.

Solve for x.

23. $\log_6 x = 2$ **24.** $\log_5 x = 3$ **25.** $\log_{16} x = \frac{1}{2}$ **26.** $\log_8 x = 2.5$

27. $\log_9 x = \frac{3}{2}$ **28.** $\log_{1/4} x = -\frac{1}{2}$ **29.** $\log_x 64 = 3$ **30.** $\log_x 8 = -1$

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10–5 Laws of Logarithms

Objective: To learn and apply the basic properties of logarithms.

Vocabulary

Laws of Logarithms Let b be the base of a logarithmic function ($b > 0$, $b \neq 1$). Let M and N be positive numbers.

1. $\log_b MN = \log_b M + \log_b N$
2. $\log_b \frac{M}{N} = \log_b M - \log_b N$
3. $\log_b M^k = k \log_b M$

Example 1 Express $\log_5 M^4N^2$ in terms of $\log_5 M$ and $\log_5 N$.

Solution

$\log_5 M^4N^2 = \log_5 M^4 + \log_5 N^2$ Use Law 1.

$= 4 \log_5 M + 2 \log_5 N$ Use Law 3.

Example 2 Express $\log_3 \sqrt[3]{\frac{M^2}{N}}$ in terms of $\log_3 M$ and $\log_3 N$.

Solution

$\log_3 \sqrt[3]{\frac{M^2}{N}} = \log_3 \left(\frac{M^2}{N}\right)^{1/3} = \frac{1}{3} \log_3 \frac{M^2}{N}$ Use Law 3.

$= \frac{1}{3} (\log_3 M^2 - \log_3 N)$ Use Law 2.

$= \frac{1}{3} (2 \log_3 M - \log_3 N)$ Use Law 3.

$= \frac{2}{3} \log_3 M - \frac{1}{3} \log_3 N$ Multiply.

Express each logarithm in terms of $\log_2 M$ and $\log_2 N$.

1. $\log_2 M^4N^3$
2. $\log_2 (MN)^3$
3. $\log_2 M \sqrt[3]{N}$
4. $\log_2 \sqrt{M^3N}$
5. $\log_2 \frac{M^5}{N^6}$
6. $\log_2 \left(\frac{N}{M}\right)^4$
7. $\log_2 \sqrt{\frac{M}{N}}$
8. $\log_2 \sqrt[3]{\frac{M}{N^4}}$

Example 3 Express as a single logarithm.

a. $\log_2 M + 2 \log_2 N$

b. $\frac{1}{2} \log_2 M - 3 \log_2 N$

Solution

a. $\log_2 M + 2 \log_2 N =$

$\log_2 M + \log_2 N^2 =$ (Law 3)

$\log_2 MN^2$ (Law 1)

b. $\frac{1}{2} \log_2 M - 3 \log_2 N =$

$\log_2 \sqrt{M} - \log_2 N^3 =$ (Law 3)

$\log_2 \frac{\sqrt{M}}{N^3}$ (Law 2)

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10–5 *Laws of Logarithms* *(continued)*

Express as a single logarithm.

9. $2 \log_2 M + 4 \log_2 N$ **10.** $\log_5 N - 3 \log_5 M$ **11.** $\log_{10} M + 5 \log_{10} N$

12. $7 \log_2 N - \log_2 M$ **13.** $\frac{1}{3} \log_3 M + 4 \log_3 N$ **14.** $\frac{3}{2} \log_4 M - \log_4 N$

Example 4 If $\log_{10} 4 = 0.60$ and $\log_{10} 3 = 0.48$, find:

a. $\log_{10} 40$ **b.** $\log_{10} 36$ **c.** $\log_{10} \frac{1}{\sqrt{3}}$

Solution

a. $\log_{10} 40 = \log_{10} (4 \times 10)$ — $40 = 4 \times 10$

$= \log_{10} 4 + \log_{10} 10$ — Use Law 1.

$= 0.60 + 1 = 1.60$ — Substitute. ($\log_{10} 10 = 1$)

b. $\log_{10} 36 = \log_{10} (3^2 \times 4)$ — $36 = 9 \times 4 = 3^2 \times 4$

$= \log_{10} 3^2 + \log_{10} 4$ — Use Law 1.

$= 2 \log_{10} 3 + \log_{10} 4$ — Use Law 3.

$= 2(0.48) + 0.60 = 1.56$ — Substitute.

c. $\log_{10} \frac{1}{\sqrt{3}} = \log_{10} 1 - \log_{10} \sqrt{3}$ — Use Law 2.

$= \log_{10} 1 - \log_{10} 3^{1/2}$ — Write the radical in exponential form.

$= \log_{10} 1 - \frac{1}{2} \log_{10} 3$ — Use Law 3.

$= 0 - \frac{1}{2}(0.48) = -0.24$ — Substitute. ($\log_{10} 1 = 0$)

If $\log_{10} 4 = 0.60$ and $\log_{10} 3 = 0.48$ (accurate to two decimal places), find the following.

15. $\log_{10} 12$ **16.** $\log_{10} 16$ **17.** $\log_{10} 27$ **18.** $\log_{10} 48$

19. $\log_{10} \sqrt{3}$ **20.** $\log_{10} 2$ **21.** $\log_{10} \frac{1}{3}$ **22.** $\log_{10} \frac{2}{3}$

23. $\log_{10} \frac{30}{4}$ **24.** $\log_{10} 400$ **25.** $\log_{10} \frac{1}{4000}$ **26.** $\log_{10} \sqrt{\frac{3}{4}}$

Mixed Review Exercises

Solve.

1. $3^{x-1} = 9^{x+3}$ **2.** $\sqrt{x} + 6 = x$ **3.** $\log_x 27 = 3$ **4.** $2(x - 5) = 3x + 6$

5. $(x + 3)^2 = 7$ **6.** $1 - \frac{4}{x} = \frac{1}{x - 3}$ **7.** $2^x = \frac{\sqrt{2}}{8}$ **8.** $x^3 + 3x^2 - 4x - 12 = 0$

If $f(x) = 2x^2 + 1$ and $g(x) = \sqrt{x}$, find each of the following.

9. $f(3)$ **10.** $g(16)$ **11.** $f(g(4))$ **12.** $g(f(2))$ **13.** $f(g(a))$ **14.** $g(f(a))$

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10–6 Applications of Logarithms

Objective: To use common logarithms to solve equations involving powers and to evaluate logarithms with any given base.

Vocabulary

Common logarithms Logarithms with base 10. The base 10 is usually not written. Example: log 3 means $\log_{10} 3$.

Characteristic Using scientific notation, the common logarithm of a number can be written as the sum of an integer and a nonnegative number less than 1. The characteristic is the integral part of a logarithm.

Mantissa The decimal, or fractional, part of a logarithm.

Antilogarithm If $\log y = a$, the number y is sometimes called the antilogarithm of a.

CAUTION If you will be using a calculator to find logarithms, skip over Examples 1 and 2.

Example 1 **Using a Table of Logarithms** The table below gives the common logarithms of some numbers between 1 and 10 rounded to four decimal places. The decimal point is omitted. To find an approximation for the logarithm of a number x between 0 and 10, find the first two digits of the number in the column marked N, and the third digit in the row to the right of N. The number in the spot where that row and column intersect is the logarithm of x.

To find log 3.27, find 32 in the column under N, and 7 in the row to the right of N. The value where row 32 intersects column 7 is 5145. Thus, log 3.27 = 0.5145.

N	0	1	2	3	4	5	6	7	8	9
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	**5145**	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428

Example 2 Find each logarithm using the table above: **a.** log 3400 **b.** log 0.0319

Solution Write each number in scientific notation and use the laws of logarithms.

a. $\log 3400 = \log(3.40 \times 10^3) = \log 3.40 + 3 \log 10 = 0.5315 + 3$

$\therefore \log 3400 = 3.5315$. The mantissa is 0.5315. The characteristic is 3.

b. $\log 0.0319 = \log(3.19 \times 10^{-2}) = \log 3.19 - 2 \log 10 = 0.5038 - 2$

$\therefore \log 0.0319 = -1.4962$. The mantissa is 0.5038. The characteristic is -2.

Use a calculator or the table above to find each logarithm.

1. log 32.5 **2.** log 33,200 **3.** log 0.348 **4.** log 0.00307

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10-6 Applications of Logarithms *(continued)*

Example 3 Find y to three significant digits if: **a.** $\log y = 0.4983$ **b.** $\log y = 3.4812$

Using Tables

a. Find the value closest or equal to 0.4983 among the entries in the table in Example 1. This is the entry in the row labeled 31 and in the column labeled 5.
$\therefore y = 3.15$

b. First find the number that has 0.4812 as its mantissa. The table entry with mantissa closest to 0.4812 (0.4814) is 3.03. Working backwards you have:

$$\log y = 3.4812 = 0.4812 + 3 = \log 3.03 + \log 10^3 = \log (3.03 \times 10^3)$$

$$\therefore y = 3.03 \times 10^3 = 3030$$

Using a Calculator On some calculators you find the antilogarithm by using the inverse function key with the logarithm key. On others, you can use the 10^x key as shown.

a. If $\log y = 0.4983$, then $y = 10^{0.4983} = 3.15$.
b. If $\log y = 3.4812$, then $y = 10^{3.4812} = 3030$.

Use a calculator or the table in Example 1 to find x to three significant digits.

5. $\log x = 0.5378$ **6.** $\log x = 0.5009$ **7.** $\log x = 1.507$ **8.** $\log x = 2.51$

Example 4 Solve $(3.2)^x = 3$. First give the solution in calculation-ready form, and then to three significant digits.

Solution

$(3.2)^x = 3$	Take the logarithm of both sides.
$\log (3.2)^x = \log 3$	Use laws of logarithms to simplify.
$x \log 3.2 = \log 3$	Solve for x.
$x = \dfrac{\log 3}{\log 3.2}$	**(calculation-ready form)**
$x = \dfrac{0.4771}{0.5051}$	Find log 3 and log 3.2 with a calculator or the given table.
$= 0.945$	**(to three significant digits)**

Solve each equation. Give the solution (a) in calculation-ready form, and (b) to 3 significant digits.

9. $34^x = 30$ **10.** $3.1^x = 300$ **11.** $33^{-x} = 3$ **12.** $(3.09)^{2x} = 34$

Example 5 Solve $x^{3/5} = 32$. Give your answer to three significant digits.

Solution

1. Raise each side to the $\frac{5}{3}$ power. You find that $x = 32^{5/3}$.
2. To simplify $32^{5/3}$ do one of the following:
   (a) Use a calculator and the y^x key to obtain $x = 323$.
   (b) Use logarithms to obtain $\log x = \frac{5}{3} \log 32 = 2.5086$.
   Then find the antilogarithm: $x = 323$.

Solve.

13. $x^{3/5} = 31$

14. $x^{8/9} = 3$

15. $\sqrt[7]{x^3} = 32$

16. $\sqrt[3]{x^2} = 10$

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10–7 Problem Solving: Exponential Growth and Decay

Objective: To use exponential and logarithmic functions in growth and decay problems.

Vocabulary

Exponential growth A quantity exhibits exponential growth when its size after a period of time can be modeled by an exponential function whose base is greater than one.

Compound interest formula If an amount P (called the *principal*) is invested at an annual interest rate r compounded n times a year, then in t years the investment will grow to an amount A given by $A = P\left(1 + \frac{r}{n}\right)^{nt}$ where r is expressed as a decimal.

Doubling-time growth formula If a population of size N_0 doubles every d years (or hours, or days, or any other unit of time), then the number N in the population at time t is given by $N = N_0 \cdot 2^{t/d}$.

Example 1

a. One hundred dollars is invested at 7.8% interest compounded semiannually (twice a year). Determine how much the investment is worth after 3 years.

b. The value of a new \$300 VCR decreases 10% per year. Find its value after 2 years.

Solution

a. Use the compound interest formula. Let $P = 100$, $r = 0.078$, and $n = 2$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 100\left(1 + \frac{0.078}{2}\right)^{2t} = 100(1 + 0.039)^{2t}$$

Since $t = 3$, $A = 100(1 + 0.039)^{2(3)} = 100(1.039)^6 \approx 100(1.26) = 126.$

$\therefore$ the investment is worth \$126 after 3 years.

Note: To evaluate $(1.039)^6$ without a calculator, write $x = (1.039)^6$, take the logarithm of both sides, and solve for x.

b. Modify the compound interest formula. Replace P with the value of a new VCR: $V_0 = 300$. Let $r = -0.10$ (since the value *decreases* over time), and $n = 1$.

$$A = V_0\left(1 + \frac{r}{n}\right)^{nt} = 300(1 - 0.10)^t, \text{ or } 300(0.9)^t$$

Since $t = 2$, value $= 300(0.9)^2 = 300(0.81) = 243.$

$\therefore$ the value of the VCR after 2 years is \$243.

Solve. Give answers to the nearest dollar. A calculator may be helpful.

1. One thousand dollars is invested at 8% interest compounded annually. Determine how much the investment is worth after:
   a. 1 year **b.** 4 years **c.** 7 years
2. Redo Problem 1, assuming that the interest is compounded quarterly (four times a year).
3. The value of a new \$500 television decreases 10% per year. Find its value after:
   a. 1 year **b.** 5 years **c.** 10 years
4. The value of a new \$400 bicycle decreases 20% per year. Find its value after:
   a. 1 year **b.** 2 years **c.** 6 years

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10–7 Problem Solving: Exponential Growth and Decay *(continued)*

Vocabulary

Exponential decay A quantity exhibits exponential decay when its size after a period of time can be modeled by an exponential function whose base is between 0 and 1.

Half-life The amount of time it takes for exactly half of a radioactive substance to decay.

Half-life decay formula If an amount N_0 has a half-life h, then the amount remaining at time t is $N = N_0\left(\frac{1}{2}\right)^{t/h}$.

Example 2 A certain bacteria population doubles in size every 16 hours. By how much will it grow in 4 days?

Solution Use the doubling-time growth formula. Let $t = 96$ hours (4 days) and let the doubling time $d = 16$ hours.

$$N = N_0 \cdot 2^{t/d} = N_0 \cdot 2^{96/16} = N_0 \cdot 2^6 = 64N_0$$

$\therefore$ the population grows by a factor of 64 in 4 days.

Example 3 The half-life of carbon 14 (C-14) is approximately 5730 years. Determine how much of 15.0 mg of this substance will remain after 2500 years.

Solution Use the half-life decay formula. Let $N_0 = 15$, $h = 5730$, and $t = 2500$.

$$N = N_0\left(\frac{1}{2}\right)^{t/h} = 15\left(\frac{1}{2}\right)^{2500/5730}$$

If you have a calculator, use the y^x key to obtain N. Otherwise, take the logarithm of both sides and use Table 3 on pages 812 and 813 of your textbook to obtain $\log N = 1.0448$. Then find the antilogarithm. In either case, you will find that 11.1 mg of this substance will remain after 2500 years (to three significant digits).

Solve. Give answers to three significant digits. A calculator may be helpful.

5. The population of a certain country doubles in size every 60 years. The population is now N_0. Find its size, in terms of N_0, in:
a. 120 years **b.** 180 years **c.** y years

6. The half-life of carbon 14 is approximately 5730 years. Determine how much of 1000 kg of this substance will remain after 8000 years.

7. A radioactive substance has a half-life of approximately 18 hours. About how much of a 10 g sample will remain after 4 days?

Mixed Review Exercises

If the domain of each function is $D = \{1, 2, 4\}$, find the range.

1. $h(x) = \sqrt{x} - 1$ **2.** $F(x) = \log_2 x$ **3.** $G(x) = |x - 5|$ **4.** $H(x) = (\sqrt{3})^x$

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10–8 The Natural Logarithm Function

Objective: To define and use the natural logarithm function.

Vocabulary

Natural logarithm function The logarithm function with base e, where e is an irrational number approximately equal to 2.71828. The natural logarithm is used extensively in advanced science and mathematics.

Symbol **ln** x (natural logarithm of x, or $\log_e x$)

CAUTION You will have to be familiar with the properties of logarithms found in Lesson 10–4, and the laws of logarithms found in Lesson 10–5 in order to do the exercises in this lesson. These properties and laws apply to natural logarithms just as they do to logarithms with other bases.

Example 1

Exponential Form		*Logarithmic Form*
$e^{2.64} = 14$	means	$\ln 14 = 2.64$
$e^{3.09} = 22$	means	$\ln 22 = 3.09$
$e^{-1.1} = \frac{1}{3}$	means	$\ln \frac{1}{3} = -1.1$
$\sqrt[5]{e} = 1.22$	means	$\ln 1.22 = \frac{1}{5}$

Write each equation in exponential form.

1. $\ln 10 = 2.30$ **2.** $\ln 50 = 3.91$ **3.** $\ln \frac{1}{4} = -1.39$ **4.** $\ln \frac{1}{e^3} = -3$

Write each equation in logarithmic form.

5. $e^4 = 54.6$ **6.** $e^9 = 8103$ **7.** $e^{1/4} = 1.28$ **8.** $\sqrt{e} = 1.65$

Example 2 Simplify: **a.** $\ln e^5$ **b.** $\ln\frac{1}{e^3}$ **c.** $e^{\ln 3}$

Solution

a. $\ln e^5 = 5 \ln e = 5 \cdot 1 = 5$

b. $\ln\frac{1}{e^3} = \ln e^{-3} = -3 \ln e = -3 \cdot 1 = -3$

c. $e^{\ln 3} = 3$ A property of logarithms: $b^{\log_b N} = N$

Simplify. If the expression is undefined, say so.

9. $\ln e^4$ **10.** $\ln e^7$ **11.** $\ln\frac{1}{e^5}$ **12.** $\ln \sqrt{e}$

13. $\ln 1$ **14.** $\ln(-1)$ **15.** $e^{\ln 1.2}$ **16.** $e^{\ln\sqrt{3}}$

10–8 The Natural Logarithm Function *(continued)*

Example 3 Write as a single logarithm: **a.** $\ln 2 + \ln 5$ **b.** $2 \ln 3 - \ln 4 + 1$

Solution **a.** $\ln 2 + \ln 5 = \ln (2 \cdot 5)$
$= \ln 10$

b. $2 \ln 3 - \ln 4 + 1 = \ln 3^2 - \ln 4 + \ln e$
$= \ln \left(\frac{3^2}{4} \cdot e\right) = \ln \frac{9e}{4}$

Write as a single logarithm.

17. $\ln 3 + \ln 7$ **18.** $\ln 12 - \ln 3$ **19.** $\ln 11 + \frac{1}{2} \ln 4$ **20.** $3 \ln 3 - \ln 2 + 2$

Example 4 Solve: **a.** $\ln \frac{1}{x} = 4$ **b.** $\ln (x + 3) = 1$

Solution Write in exponential form. Then solve for x.

a. $\ln \frac{1}{x} = 4$
$e^4 = \frac{1}{x}$
$x = \frac{1}{e^4}$
$x = e^{-4}$

b. $\ln (x + 3) = 1$
$e^1 = x + 3$
$e - 3 = x$
$x = e - 3$

Solve for x. Leave answers in terms of e.

21. $\ln x = 5$ **22.** $\ln \frac{1}{x} = 3$ **23.** $\ln (x - 2) = 2$ **24.** $\ln \sqrt{x + 5} = 1$

Example 5 Solve: **a.** $e^x = 7$ **b.** $e^{x + 3} = 8$

Solution Take the natural logarithm of both sides of the equation. Then solve for x.

a.
$e^x = 7$
$\ln e^x = \ln 7$
$x \cdot \ln e = \ln 7$
$x \cdot 1 = \ln 7$
$x = \ln 7$

b.
$e^{2x + 8} = 9$
$\ln e^{2x + 8} = \ln 9$
$(2x + 8) \cdot \ln e = \ln 9$
$(2x + 8) \cdot 1 = \ln 9$
$2x + 8 = \ln 9$
$2x = -8 + \ln 9$
$x = \frac{-8 + \ln 9}{2}$, or $-4 + \ln 3$

Solve for x. Leave answers in terms of natural logarithms.

25. $e^x = 5$ **26.** $e^{-x} = 2$ **27.** $e^{2x} = 36$ **28.** $e^{x-4} = 3$

29. $e^{3x-6} = 27$ **30.** $\sqrt{e^x} = 4$ **31.** $e^{-3x} = \frac{1}{8}$ **32.** $(e^x)^4 = 81$