NAME ______________________ DATE ____________

12 Triangle Trigonometry

12–1 Angles and Degree Measure

Objective: To use degrees to measure angles.

Vocabulary

Degree A measure of rotation. One complete revolution measures 360 degrees.

Angle A figure formed by two rays that have the same endpoint.

Directed angle An angle generated by the rotation of a ray (the *initial side*) onto another ray (the *terminal side*).

Positive angle An angle generated by a counterclockwise rotation.

Negative angle An angle generated by a clockwise rotation.

Standard position An angle is in standard position when its initial side coincides with the positive x-axis.

Quadrantal angle An angle whose terminal side lies on a coordinate axis.

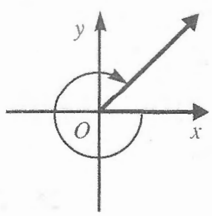
Coterminal angles Two angles whose terminal sides coincide when the angles are in standard position.

Symbols $1°$ (1 degree $= \frac{1}{360}$ revolution) $1'$ (1 minute $= \frac{1}{60}°$) $1''$ (1 second $= \frac{1}{3600}°$)

Example 1 **a.** Sketch $-315°$ in standard position. Indicate its rotation by a curved arrow and classify the angle by the quadrant containing its terminal side.

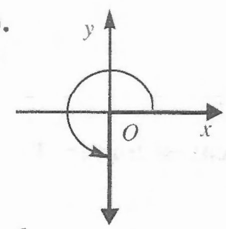
b. Sketch $\frac{3}{4}$ of a counterclockwise revolution and find its measure.

Solution **a.**



Quadrant I

b.



$\frac{3}{4} \times 360° = 270°$

Sketch each angle in standard position. Indicate its rotation by a curved arrow. Classify each angle by its quadrant. If the angle is a quadrantal angle, say so.

1. $60°$ **2.** $-60°$ **3.** $120°$ **4.** $-120°$ **5.** $200°$

6. $-200°$ **7.** $330°$ **8.** $-390°$ **9.** $450°$ **10.** $-820°$

Sketch in standard position the angle described and find its measure.

11. $\frac{1}{6}$ of a counterclockwise revolution **12.** $\frac{2}{3}$ of a clockwise revolution

13. $\frac{3}{4}$ of a counterclockwise revolution **14.** $1\frac{1}{8}$ of a clockwise revolution

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–1 Angles and Degree Measure *(continued)*

Example 2 **a.** Write a formula for the measure of all angles coterminal with a $60°$ angle.

b. Use the formula to find two positive angles and two negative angles that are coterminal with a $60°$ angle.

Solution **a.** $60° + n \cdot 360°$, where n is an integer.

b. To find the measure of the angles, let $n = 1, 2, -1, -2$.

$60° + 1(360°) = 420°$ $\quad$ $60° + 2(360°) = 780°$

$60° + (-1)(360°) = -300°$ $\quad$ $60° + (-2)(360°) = -660°$

For Exercises 15–22: (a) Write a formula for the measures of all angles coterminal with the given angle. (b) Use the formula to find two angles (one positive and one negative) that are coterminal with the given angle.

15. $40°$ **16.** $120°$ **17.** $-60°$ **18.** $-235°$

19. $450°$ **20.** $210°$ **21.** $900°$ **22.** $-720°$

Example 3 **a.** Express $28°\ 12'\ 38''$ in decimal degrees.

b. Express $48.21°$ in degrees, minutes, and seconds to the nearest second.

Solution Use these facts: $1' = \left(\frac{1}{60}\right)^\circ$ and $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)^\circ$

a. $28°\ 12'\ 38'' = 28° + \left(\frac{12}{60}\right)^\circ + \left(\frac{38}{3600}\right)^\circ = 28° + 0.2° + 0.011° = 28.211°$

b. $48.21° = 48° + (0.21 \times 60)' = 48° + 12.6' = 48° + 12' + (0.6 \times 60)''$
$= 48° + 12' + 36'' = 48°\ 12'\ 36''$

Express in decimal degrees to the nearest hundredth of a degree.

23. $16°\ 40'$ **24.** $49°\ 27'$ **25.** $83°\ 52'$ **26.** $28°\ 45'$

Express in degrees, minutes, and seconds to the nearest second.

27. $26.52°$ **28.** $39.45°$ **29.** $85.39°$ **30.** $55.73°$

Mixed Review Exercises

Find the third term in the expansion of each binomial.

1. $(x - 3y)^4$ **2.** $(x + \sqrt{2y})^6$ **3.** $(x^3 + y^2)^{10}$

Simplify.

4. $\sqrt{-5} \cdot \sqrt{-20}$ **5.** $\log_2 28 - \log_2 7$ **6.** $(9^{3/4} - 9^{5/4})^2$

7. $\ln \sqrt{e}$ **8.** $(5 + 2i)(3 - 4i)$ **9.** $\sqrt{27} - \sqrt{12} + \sqrt{48}$

Copyright © by Houghton Mifflin Company. All rights reserved.

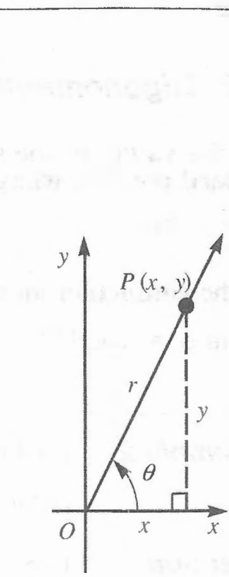
NAME ______________________________ DATE ______________

12–2 Trigonometric Functions of Acute Angles

Objective: To define trigonometric functions of acute angles.

Vocabulary

Trigonometric functions For an acute angle θ in standard position, with a point $P(x, y)$, other than the origin, on the terminal side and r = the distance OP:



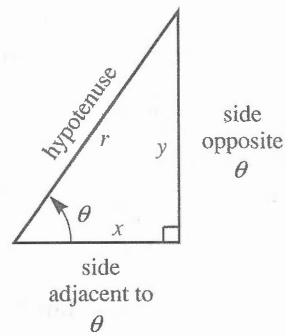
Sine (sin)	**Cosine (cos)**	**Tangent (tan)**
$\sin\theta = \dfrac{y}{r}$	$\cos\theta = \dfrac{x}{r}$	$\tan\theta = \dfrac{y}{x}$

Cotangent (cot) The cotangent of an angle θ is the reciprocal of $\tan\theta$.

Secant (sec) The secant of an angle θ is the reciprocal of $\cos\theta$.

Cosecant (csc) The cosecant of an angle θ is the reciprocal of $\sin\theta$.

Formulas Given a right triangle with sides of lengths x, y, and r, and angle θ opposite side y, then:



$$\sin\theta = \frac{y}{r} = \frac{\text{length of side opposite } \theta}{\text{length of the hypotenuse}} \qquad \csc\theta = \frac{r}{y} \text{ or } \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{length of side adjacent to } \theta}{\text{length of the hypotenuse}} \qquad \sec\theta = \frac{r}{x} \text{ or } \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta} \qquad \cot\theta = \frac{x}{y} \text{ or } \frac{1}{\tan\theta}$$

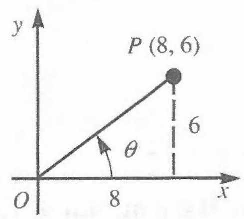
Cofunction identities in right triangle *ABC* Given complementary angles A and B (angles whose sum is 90°), the following identities hold:

$\sin A = \cos B$	$\tan A = \cot B$	$\sec A = \csc B$
$\cos A = \sin B$	$\cot A = \tan B$	$\csc A = \sec B$

Example: If $\sin A = \cos 60°$, then A = the complement of $60° = 30°$.

Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, where $\sin^2\theta$ means $(\sin\theta)^2$, and $\cos^2\theta$ means $(\cos\theta)^2$.

Example 1 Find the values of the six trigonometric functions of an angle in standard position whose terminal side passes through $P(8, 6)$.



Solution Sketch θ in standard position.
By the Pythagorean theorem $r^2 = x^2 + y^2$:

$$r^2 = 8^2 + 6^2 = 100$$
$$r = 10$$

$$\sin\theta = \frac{y}{r} = \frac{6}{10} = \frac{3}{5} \qquad \cos\theta = \frac{x}{r} = \frac{8}{10} = \frac{4}{5} \qquad \tan\theta = \frac{y}{x} = \frac{6}{8} = \frac{3}{4}$$

$$\csc\theta = \frac{r}{y} = \frac{10}{6} = \frac{5}{3} \qquad \sec\theta = \frac{r}{x} = \frac{10}{8} = \frac{5}{4} \qquad \cot\theta = \frac{x}{y} = \frac{8}{6} = \frac{4}{3}$$

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ____________

12–2 Trigonometric Functions of Acute Angles *(continued)*

Find the values of the six trigonometric functions of an angle θ in standard position whose terminal side passes through point P.

1. $P(4, 3)$ **2.** $P(3, 3)$ **3.** $P(12, 5)$ **4.** $P(1, 4)$

Use the cofunction identities to find the measure of the acute angle ϕ.

5. $\sin \phi = \cos 12°$ **6.** $\cos \phi = \sin 65°$ **7.** $\tan \phi = \cot 45°$ **8.** $\csc \phi = \sec 73°$

Example 2 Find $\sin \theta$ and $\tan \theta$ if θ is an acute angle and $\cos \theta = \frac{1}{2}$.

Solution Use the Pythagorean identity.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin^2 \theta + \frac{1}{4} = 1$$

$$\sin^2 \theta = \frac{3}{4}$$

$$\sin \theta = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \sqrt{3}$$

Complete the table. In each case, θ is an acute angle.

	$\sin \theta$	$\cos \theta$	$\tan \theta$
9.	$\frac{1}{3}$	?	?
10.	?	$\frac{5}{6}$	?
11.	$\frac{3}{4}$	?	?
12.	?	$\frac{\sqrt{7}}{3}$	?
13.	$\frac{\sqrt{85}}{11}$	?	?

Example 3 Use the diagram to find the lengths of side $\overline{BC}$ and side $\overline{AB}$.

Solution Use the table on page 558 of your textbook to find tan 45° and cos 45°.

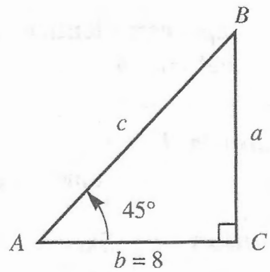
$$\tan 45° = \frac{BC}{AC} \qquad \cos 45° = \frac{AC}{AB}$$

$$1 = \frac{a}{8} \qquad \frac{\sqrt{2}}{2} = \frac{8}{c}$$

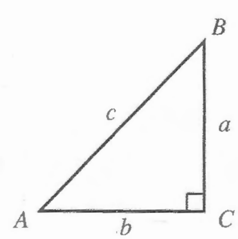
$$a = 8 \qquad \sqrt{2}c = 16$$

$$\therefore \text{ side } BC = 8 \qquad c = \frac{16}{\sqrt{2}} = 8\sqrt{2}$$

$$\therefore \text{ side } AB = 8\sqrt{2}$$



Use the diagram at the right to find the lengths of the sides and the measures of the angles that are not given. Leave your answers in simplest radical form. You may wish to use the table on page 558 of the text.



14. $a = 8, \angle A = 30°$ **15.** $b = 5, \angle A = 60°$ **16.** $c = 12, \angle A = 45°$

17. $c = 50, \angle B = 30°$ **18.** $a = 4, c = 8$ **19.** $a = 2, b = 2$

Copyright © by Houghton Mifflin Company. All rights reserved.

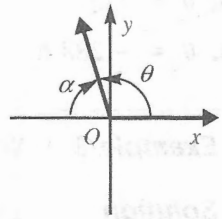
NAME ______________________________ DATE ______________

12–3 Trigonometric Functions of General Angles

Objective: To define trigonometric functions of general angles.

Vocabulary

Reference angle If θ is not a quadrantal angle, there is a unique acute angle α, corresponding to θ, such that either $\theta + \alpha$ or $\theta - \alpha$ is an integral multiple of $180°$. α is called the reference angle of θ.
Example: If θ measures $115°$, its reference angle, α, measures $65°$.



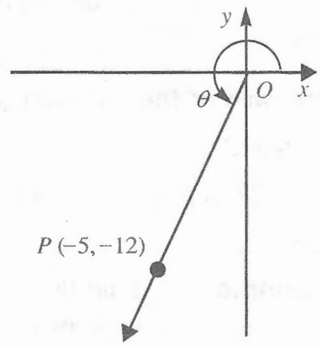
Formulas For any angle θ and any point $P(x, y)$ on the terminal side of θ:

$\sin\theta = \frac{y}{r}$ $\qquad$ $\cos\theta = \frac{x}{r}$ $\qquad$ $\tan\theta = \frac{y}{x}$, if $x \neq 0$

$\csc\theta = \frac{r}{y}$, if $y \neq 0$ $\qquad$ $\sec\theta = \frac{r}{x}$, if $x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y}$, if $y \neq 0$

Example 1 Find the values of the six trigonometric functions of an angle θ in standard position whose terminal side passes through $(-5, -12)$.

Solution First make a sketch.
Here $x = -5$ and $y = -12$.
Since r is the distance between $(0, 0)$ and $(-5, -12)$, you can find r using the distance formula.



$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$r = \sqrt{(-5 - 0)^2 + (-12 - 0)^2}$$

$$r = \sqrt{25 + 144} = \sqrt{169} = 13$$

$\sin\theta = -\frac{12}{13}$ $\qquad$ $\cos\theta = -\frac{5}{13}$ $\qquad$ $\tan\theta = \frac{-12}{-5} = \frac{12}{5}$

$\csc\theta = -\frac{13}{12}$ $\qquad$ $\sec\theta = -\frac{13}{5}$ $\qquad$ $\cot\theta = \frac{-5}{-12} = \frac{5}{12}$

Find the values of the six trigonometric functions of an angle θ in standard position whose terminal side passes through point P. If any value is undefined, say so.

1. $P(4, -3)$ $\qquad$ **2.** $P(-12, 5)$ $\qquad$ **3.** $P(-2, -2)$ $\qquad$ **4.** $P(15, -8)$ $\qquad$ **5.** $P(0, 1)$

Example 2 Find the measure of the reference angle α for each given angle θ.

a. $\theta = 155°$ $\qquad$ **b.** $\theta = 310°30'$ $\qquad$ **c.** $\theta = -125°$

Solution

a. Since θ is in quadrant II: $\alpha = 180° - \theta = 180° - 155° = 25°$

b. Since θ is in quadrant IV:
$\alpha = 360° - \theta = 360° - 310°30' = 359°60' - 310°30' = 49°30'$

c. First find the positive angle that is coterminal with $-125°$:
$$-125° + 360° = 235°$$
Since $235°$ is in quadrant III: $\alpha = \theta - 180° = 235° - 180° = 55°$

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–3 Trigonometric Functions of General Angles *(continued)*

Find the measure of the reference angle α of the given angle θ.

6. $\theta = 250°$ **7.** $\theta = 172°$ **8.** $\theta = -200°$ **9.** $\theta = -142°$

10. $\theta = 730°$ **11.** $\theta = 480°$ **12.** $\theta = 98.2°$ **13.** $\theta = -166.2°$

14. $\theta = -288.6°$ **15.** $\theta = 149°20'$ **16.** $\theta = 243°15'$ **17.** $\theta = 300°50'$

Example 3 Write $\sin 230°$ as a function of an acute angle.

Solution The reference angle of 230° is $230° - 180° = 50°$.
Since 230° is in the third quadrant, its sine is negative (see note below).
$\therefore \sin 230° = -\sin 50°$.

Note: The signs of the trigonometric functions vary with the quadrants. All values are positive in the first quadrant. Sine and cosecant are positive in the second quadrant. Tangent and cotangent are positive in the third quadrant. Cosine and secant are positive in the fourth quadrant.

Write each of the following as a function of an acute angle.

18. $\sin 312°$ **19.** $\cos 215°$ **20.** $\tan(-163°)$ **21.** $\sin(-29°)$

22. $\sec 276.8°$ **23.** $\csc(-198.3°)$ **24.** $\cot 312°15'$ **25.** $\cos 259°12'$

Example 4 Find the exact values of the six trigonometric functions of 225°. Use the table of exact trigonometric values found in Lesson 12-2 of your textbook.

Solution The reference angle of 225° is 45°.

Since 225° is in the third quadrant, only the tangent and cotangent are positive.

$\sin 225° = -\sin 45° = -\frac{\sqrt{2}}{2}$ $\csc 225° = -\csc 45° = -\sqrt{2}$

$\cos 225° = -\cos 45° = -\frac{\sqrt{2}}{2}$ $\sec 225° = -\sec 45° = -\sqrt{2}$

$\tan 225° = \tan 45° = 1$ $\cot 225° = \cot 45° = 1$

Find the exact values of the six trigonometric functions of each angle.

26. 120° **27.** −135° **28.** 210° **29.** −225° **30.** 420°

Mixed Review Exercises

Find the zeros of each function. If the function has no zeros, say so.

1. $f(x) = 2x + 6$ **2.** $g(x) = \log(x - 2)$ **3.** $h(x) = 6x^2 + x - 2$

4. $F(x) = \frac{x^3 - 9x}{x^2 + 2}$ **5.** $G(x) = x^2 + 3x + 8$ **6.** $H(x) = 2^x$

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–4 Values of Trigonometric Functions

Objective: To use a calculator or trigonometric tables to find values of trigonometric functions.

Example 1 Find each function value to four significant digits: **a.** cos 29°40′ **b.** csc 57.7°

Solution 1 **Using a Calculator**

a. Some calculators operate only with decimal degrees, so you may need to change 29°40′ to decimal degrees. Divide 40′ by 60 (60′ = 1°).

$$\cos 29°40' = \cos\left(29 + \frac{40}{60}\right)^\circ = \cos 29.67°$$

$$\cos 29.67° = 0.8688908$$

$\therefore$ to four significant digits cos 29.67° = 0.8689.

b. If your calculator does not have keys labeled sec, csc, and cot, you may need to use reciprocal functions and the reciprocal key on your calculator.

$$\csc 57.7° = \frac{1}{\sin 57.7°} = \frac{1}{0.8452618} = 1.1830654$$

$\therefore$ to four significant digits csc 57.7° = 1.183.

Solution 2 **Using Tables**

a. In Table 5 of your textbook look *down* the column under "cos θ" until you find the entry opposite 29°40′ at the left. You should find "0.8689."

$\therefore$ cos 29°40′ = 0.8689.

b. In Table 4 of the text look *up* the column over "csc θ" until you find the entry opposite 57.7° at the right. You should find "1.183."

$\therefore$ csc 57.7° = 1.183.

Example 2 Find cos 161.3° and sin 161.3°.

Solution 1 **Using a Calculator**

You may obtain the values directly. cos 161.3° = −0.9472; sin 161.3° = 0.3206

Solution 2 **Using Tables**

The reference angle for 161.3° is 180° − 161.3° = 18.7°. Since 161.3° is a second-quadrant angle, its cosine is negative and its sine is positive.

$\therefore$ cos 161.3° = −cos 18.7° = −0.9472; sin 161.3° = sin 18.7° = 0.3206

Find each function value to four significant digits.

1. tan 51.3°	**2.** cos 54.7°	**3.** sin 38.6°	**4.** cot 12.8°
5. sec 13.3°	**6.** csc 41.5°	**7.** tan 43°10′	**8.** cos 57°40′
9. sin 81°20′	**10.** cot 63°50′	**11.** sec 41°40′	**12.** csc 39°30′
13. tan 112.3°	**14.** cos 201.3°	**15.** sin 142°10′	**16.** cot 255°20′

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–4 Values of Trigonometric Functions *(continued)*

Example 3 Find $\sin 65°44'$.

Solution 1 **Using a Calculator**
You may need to change $65°44'$ to decimal degrees by dividing 44 by 60.
$\sin 65°44' = \sin 65.73° = 0.9116186$

Solution 2 **Using Tables**
Notice that $\sin 65°40' < \sin 65°44' < \sin 65°50'$.
Using a vertical arrangement for the linear interpolation, you can write:

$$\begin{array}{cc|c|cc} & & \theta & \sin\theta & & \\ \hline & & 65^\circ\ 50' & 0.9124 & & \\ 10' & & 65^\circ\ 44' & ? & & 0.0012 \\ & 4' & 65^\circ\ 40' & 0.9112 & d & \end{array}$$

Note: You must read upward to find $\sin 65°44'$ in the table.

$$\frac{d}{0.0012} = \frac{4}{10};\ d = \frac{4}{10}(0.0012) = 0.00048 \text{ or } 0.0005$$

$\sin 65°44' = 0.9112 + 0.0005 = 0.9117$

Find each function value to four significant digits.

17. $\sin 31°\ 27'$ **18.** $\cos 42°\ 53'$ **19.** $\tan 15.28°$ **20.** $\sec 51.34°$

Example 4 Find the measure of the acute angle θ to the nearest tenth of a degree, and to the nearest minute, when $\cos\theta = 0.6253$.

Solution 1 **Using a Calculator**
To find an acute angle when given one of its function values, you use the inverse function keys ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, or inv sin, inv cos, inv tan).
If $\cos\theta = 0.6253$, then
$\theta = \text{inv cos } 0.6253 = 51.29579°$
$\therefore$ to the nearest tenth of a degree, $\theta = 51.3°$.

To convert $0.29579°$ to minutes, multiply by 60 to obtain 17.7474.
$\therefore$ to the nearest minute, $\theta = 51°18'$.

Solution 2 **Using Tables**
Reverse the process described in Example 1. Look in the cosine columns of Table 4 until you find the entry nearest 0.6253. This entry is 0.6252, and it is opposite $51.3°$ on the right. Convert to minutes as you did in Solution 1.

$\therefore$ to the nearest tenth of a degree, $\theta = 51.3°$. To the nearest minute, $\theta = 51°18'$.

Find the measure of the acute angle θ (a) to the nearest tenth of a degree, and (b) to the nearest minute.

21. $\sin\theta = 0.2763$ **22.** $\tan\theta = 1.112$

23. $\cos\theta = 0.6724$ **24.** $\csc\theta = 1.762$

25. $\sec\theta = 3.253$ **26.** $\cot\theta = 2.458$

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–5 Solving Right Triangles

Objective: To find the sides and angles of a right triangle.

Vocabulary

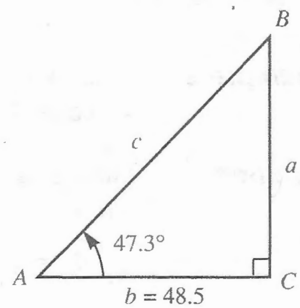
Solving a triangle Finding the values for all the sides and angles of the triangle.

Angle of elevation The angle θ by which an observer's line of sight must be raised from the horizontal to see an object.

Angle of depression The angle ϕ by which an observer's line of sight must be lowered from the horizontal to see an object.

Example 1 Solve the right triangle ABC if $\angle A = 47.3°$, $b = 48.5$, and $\angle C = 90°$. Give the lengths to three significant digits.

Solution 1 Make a diagram.



Since $\angle A + \angle B = 90°$,
$\angle B = 90° - \angle A$
$= 90° - 47.3° = 42.7°.$

$$\tan A = \frac{a}{b} \qquad \sec A = \frac{c}{b}$$

$$\tan 47.3° = \frac{a}{48.5} \qquad \sec 47.3° = \frac{c}{48.5}$$

$$a = 48.5(\tan 47.3°) \qquad c = 48.5(\sec 47.3°)$$

$$= 48.5(1.084) \qquad = 48.5(1.475)$$

$$= 52.6 \qquad = 71.5$$

$\therefore \angle B = 42.7°$, $a = 52.6$, and $c = 71.5$

Solution 2 Again $\angle B = 90° - 47.3° = 42.7°$.

$$\cot A = \frac{b}{a} \qquad \cos A = \frac{b}{c}$$

$$\cot 47.3° = \frac{48.5}{a} \qquad \cos 47.3° = \frac{48.5}{c}$$

$$a = \frac{48.5}{\cot 47.3°} \qquad c = \frac{48.5}{\cos 47.3°}$$

$$= \frac{48.5}{0.9228} = 52.6 \qquad = \frac{48.5}{0.6782} = 71.5$$

$\therefore \angle B = 42.7°$, $a = 52.6$, and $c = 71.5$

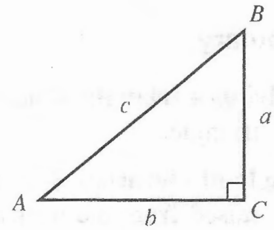
CAUTION Before you can solve a triangle given only the lengths of two sides, you must find the measure of the included angle by using the inverse of the trigonometric function relating the two sides.

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–5 *Solving Right Triangles* *(continued)*

Solve each right triangle ABC. Give lengths to three significant digits and angle measures to the nearest tenth of a degree or nearest ten minutes. You may wish to use a calculator.



1. $\angle A = 26.5°$, $c = 56$
2. $\angle B = 16.8°$, $c = 10.8$
3. $\angle B = 59.4°$, $a = 3.46$
4. $\angle A = 78.6°$, $b = 358$
5. $\angle B = 56.9°$, $b = 83.5$
6. $a = 156$, $c = 248$
7. $b = 49.6$, $c = 91.3$
8. $a = 4.60$, $b = 7.90$
9. $\angle B = 46° 20'$, $c = 125$
10. $\angle A = 25° 30'$, $a = 21.3$
11. $\angle B = 32° 50'$, $b = 46.2$
12. $\angle A = 80° 40'$, $b = 0.725$

Example 2 What is the angle of elevation of the sun when a tree 20 ft tall casts a shadow 29.3 ft long?

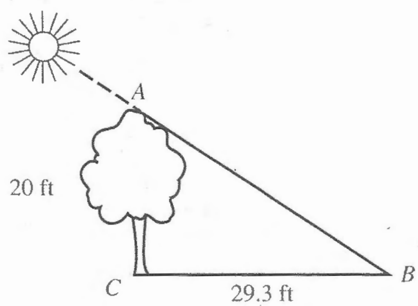
Solution Make a sketch of the problem.
$\angle B$ is the angle of elevation of the sun.

$\tan B = \dfrac{20}{29.3}$

$\tan B = 0.6826$

$\angle B = 34.3°$

$\therefore$ the angle of elevation of the sun is 34.3°.



13. What is the angle of elevation of the sun when a flagpole 10 m tall casts a shadow 12.6 m long?
14. A window washer 50 ft above the ground sees a parked car 153 ft away. What is the angle of depression from the man to the car?
15. A ladder 8 ft long is leaning against a wall. It makes an angle of 65° with the ground. How far up the wall is the top of the ladder?

Mixed Review Exercises

Give the exact values of the six trigonometric functions of each angle. If any function is not defined for the angle, say so.

1. 270°
2. 225°
3. 390°
4. −210°

Write in simplest form without negative exponents.

5. $\dfrac{x^2 - 1}{x - x^{-1}}$
6. $\sqrt{72x^6}$
7. $\dfrac{1}{x+3} + \dfrac{6}{x^2 - 9}$
8. $(-4x^3y^2)^2(3xy^2)^3$
9. $(x^{3/4})^{-4}$
10. $(2x + 3)(4x^2 - 6x + 9)$

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–6 The Law of Cosines

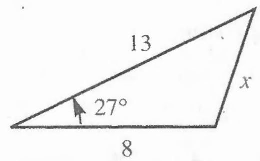
Objective: To use the law of cosines to find sides and angles of triangles.

Vocabulary

The Law of Cosines In any triangle ABC,

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$

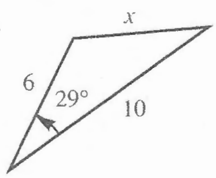
Example 1 Use the law of cosines to give an equation involving the side x of the given triangle.



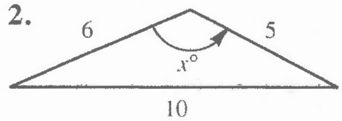
Solution $x^2 = 13^2 + 8^2 - 2(13)(8) \cos 27°$

Use the law of cosines to give an equation involving the side or angle labeled x.

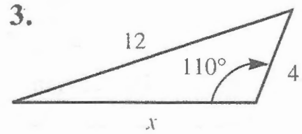
1.



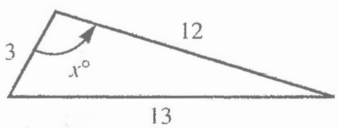
2.



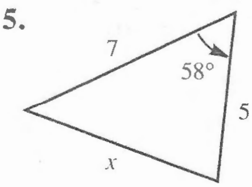
3.



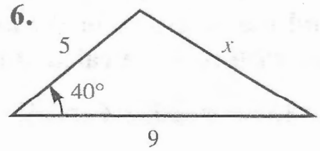
4.



5.



6.

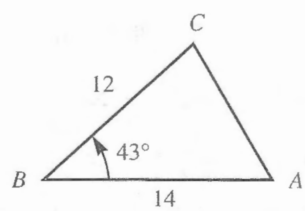


Example 2 In ΔABC, $a = 12$, $c = 14$, and $\angle B = 43°$. Find b to three significant digits.

Solution

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$b^2 = 12^2 + 14^2 - 2(12)(14) \cos 43°$$
$$= 144 + 196 - 336(0.7314)$$
$$= 340 - 245.7504$$
$$= 94.2496$$
$$b = \sqrt{94.2496} = 9.7082233$$

$\therefore$ to three significant digits, $b = 9.71$.



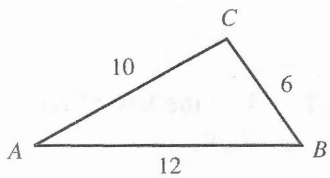
Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–6 The Law of Cosines *(continued)*

Example 3 The lengths of the three sides of triangle ABC are 6, 10, and 12. Find the smallest angle of the triangle to the nearest tenth of a degree.

Solution Make a sketch of triangle ABC. Let $a = 6$, $b = 10$, and $c = 12$. The smallest angle of the triangle is opposite the shortest side. So, angle A is the angle you must find.



Use the third form of the law of cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Substitute into the equation.

$$\cos A = \frac{10^2 + 12^2 - 6^2}{2(10)(12)}$$

$$= \frac{100 + 144 - 36}{240}$$

$$= \frac{208}{240}$$

$$= 0.8667$$

$$\angle A = 29.92°$$

$\therefore$ to the nearest tenth of a degree, $\angle A = 29.9°$.

Find the indicated part of ΔABC. Find lengths to three significant digits and the measures of the angles to the nearest tenth of a degree. You may wish to use a calculator.

7. $a = 15$, $b = 20$, $\angle C = 75°$, $c =$ _?_

8. $a = 22$, $c = 31$, $\angle B = 64.3°$, $b =$ _?_

9. $b = 54$, $c = 42$, $\angle A = 39.5°$, $a =$ _?_

10. $a = 36$, $b = 39$, $\angle C = 140°$, $c =$ _?_

11. $a = 12$, $c = 16$, $\angle B = 160°$, $b =$ _?_

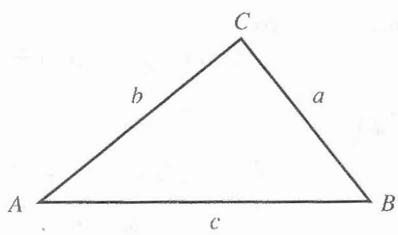
12. $a = 40$, $b = 34$, $c = 52$, $\angle C =$ _?_

13. $a = 10$, $b = 14$, $c = 16$, $\angle A =$ _?_

14. $a = 5.3$, $b = 4.7$, $c = 7.2$, $\angle B =$ _?_

15. $a = 12$, $b = 16$, $c = 19$, smallest angle = _?_

16. $a = 2.4$, $b = 3.6$, $c = 4.8$, largest angle = _?_



Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–7 The Law of Sines

Objective: To use the law of sines to find sides and angles of triangles.

Vocabulary

The Law of Sines In any triangle ABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

CAUTION When using the law of sines to find angles in triangles, be sure that the sum of the angles does not exceed 180°.

Example 1 In $\triangle ABC, \angle A = 50°$ and $a = 12$.

a. Find $\angle C$ if $c = 14$. **b.** Find $\angle C$ if $c = 9$.

Solution Use the formula $\frac{\sin A}{a} = \frac{\sin C}{c}$.

a. $\frac{\sin 50°}{12} = \frac{\sin C}{14}$

$$\sin C = \frac{14 \sin 50°}{12} = \frac{14(0.7660)}{12} = 0.8937$$

$\angle C = 63.3°$ or $\angle C = 116.7°$

Since $50° + 63.3° = 113.3°$, and $50° + 116.7° = 166.7°$, and both 113.3° and 166.7° are less than 180°, there are two solutions.

$\therefore \angle C = 63.3°$ or $\angle C = 116.7°$

b. $\frac{\sin 50°}{12} = \frac{\sin C}{9}$

$$\sin C = \frac{9 \sin 50°}{12} = \frac{9(0.7660)}{12} = 0.5745$$

$\angle C = 35.1°$ or $\angle C = 144.9°$

144.9° is not a solution because $\angle A + \angle C = 50° + 144.9° = 194.9°$, which is greater than 180°.

$\therefore \angle C = 35.1°$

Find the indicated part of $\triangle ABC$ to the nearest tenth of a degree. If there are two solutions, give both. You may wish to use a calculator.

1. $a = 6, b = 5, \angle A = 50°, \angle B = \underline{?}$
2. $a = 10, b = 13, \angle B = 75°, \angle A = \underline{?}$
3. $a = 12, c = 14, \angle C = 76°, \angle A = \underline{?}$
4. $b = 17, c = 10, \angle C = 32°, \angle B = \underline{?}$

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________ DATE ______________

12–7 The Law of Sines *(continued)*

Example 2 In $\triangle ABC$, $\angle A = 28°$ and $\angle B = 65°$.

a. Find a if $b = 12$. **b.** Find c if $a = 6$.

Solution **a.**

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 28°}{a} = \frac{\sin 65°}{12}$$

$$a = \frac{12 \sin 28°}{\sin 65°}$$

$$= \frac{12(0.4695)}{0.9063} = 6.216$$

$\therefore a = 6.22$ to three significant digits.

b. First find $\angle C$. $\angle A + \angle B + \angle C = 180°$ so $\angle C = 180° - 28° - 65° = 87°$.

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 28°}{6} = \frac{\sin 87°}{c}$$

$$c = \frac{6 \sin 87°}{\sin 28°}$$

$$= \frac{6(0.9986)}{0.4695} = 12.76$$

$\therefore c = 12.8$ to three significant digits.

Find the indicated part of $\triangle ABC$ to three significant digits. You may wish to use a calculator.

5. $a = 21, \angle A = 28°, \angle B = 62°, b = \underline{?}$
6. $c = 28, \angle A = 52°, \angle C = 72°, a = \underline{?}$
7. $b = 2.6, \angle A = 106°, \angle C = 50°, a = \underline{?}$
8. $c = 27, \angle A = 41°, \angle C = 115°, b = \underline{?}$
9. $a = 3.75, \angle B = 50°, \angle C = 110°, c = \underline{?}$
10. $b = 11.6, \angle B = 85°, \angle C = 45°, a = \underline{?}$

Mixed Review Exercises

Give the following values to four significant digits.

1. $\cos 31.5°$
2. $\tan(-75° 10')$
3. $\sin 243.6°$
4. $\csc 321° 50'$
5. $\cot 127.8°$
6. $\sec 431° 10'$

Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________________ DATE ______________

12–8 Solving General Triangles

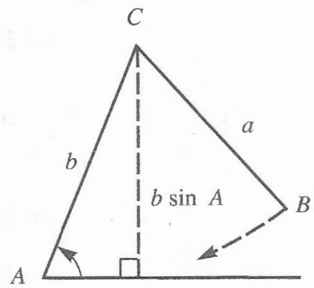
Objective: To solve any given triangle.

Vocabulary

Solving triangles The problem of solving triangles can be divided into four cases:

SSS: Given three sides
SAS: Given two sides and the included angle
SSA: Given two sides and the angle opposite one of them
ASA and AAS: Given two angles and one side

CAUTION The SSA case is sometimes called the *ambiguous* case because no triangle, one triangle, or two triangles are possible. Suppose you are given a, b, and acute $\angle A$ of ΔABC as shown at the right. Notice that the altitude from C to the side opposite $\angle C$ is $b \sin A$.

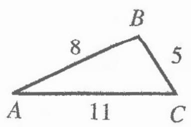


1. If $a < b \sin A$, then $\overline{BC}$ will not connect with the side opposite $\angle C$ to form a triangle.
2. If $a = b \sin A$, then ΔABC will be a right triangle.
3. If $b \sin A < a < b$, then $\overline{BC}$ will connect with the side opposite $\angle C$ in two places to form two different triangles.
4. If $a > b$, then $\overline{BC}$ will connect with the side opposite $\angle C$ in one place to form one triangle.

Example 1 **(SSS case)** Solve ΔABC if $a = 5$, $b = 11$, and $c = 8$.

Solution Use the law of cosines to find two of the angles, say $\angle A$ and $\angle B$.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{11^2 + 8^2 - 5^2}{2(11)(8)} = \frac{160}{176} = 0.9091, \text{ so } \angle A = 24.6°$$



$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{5^2 + 8^2 - 11^2}{2(5)(8)} = -\frac{32}{80} = -0.4, \text{ so } \angle B = 113.6°$$

(*Note*: Although you can also use the law of sines to find $\angle A$ and $\angle B$, you must be sure to recognize that $\angle B$ is an obtuse angle.)

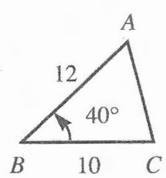
$\angle C = 180° - \angle A - \angle B = 180° - 24.6° - 113.6° = 41.8°$

$\therefore \angle A = 24.6°, \angle B = 113.6°$, and $\angle C = 41.8°$

Example 2 **(SAS case)** Solve ΔABC if $a = 10$, $c = 12$, and $\angle B = 40°$.

Solution Use the law of cosines to find the third side, b, and one of the angles, say $\angle A$.

$$b^2 = a^2 + c^2 - 2ac \cos B = 10^2 + 12^2 - 2(10)(12) \cos 40°$$
$$= 60.16, \text{ so } b = \sqrt{60.16} = 7.76$$



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{7.76^2 + 12^2 - 10^2}{2(7.76)(12)} = \frac{104.22}{186.24} = 0.5596; \angle A = 56.0°$$

$\angle C = 180° - 40° - 56.0° = 84.0°$

$\therefore b = 7.76, \angle A = 56.0°$, and $\angle C = 84.0°$

Copyright © by Houghton Mifflin Company. All rights reserved.

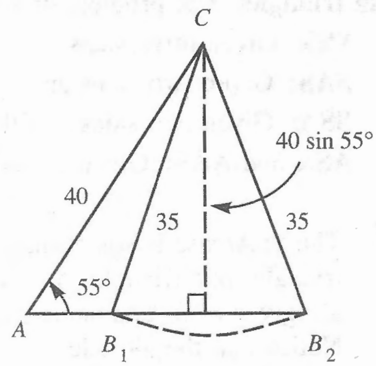
12–8 Solving General Triangles *(continued)*

Example 3 **(SSA case)** Solve $\Delta\, ABC$ if $a = 35$, $b = 40$, and $\angle A = 55°$.

Solution This is the case that may have no solution, one solution, or two solutions. Since the altitude from C to the side opposite $\angle C$ is

$$b \sin A = 40 \sin 55° = 32.8,$$

you have $32.8 < 35 < 40$ (that is, $b \sin A < a < b$), so two triangles are possible, as shown at the right.



Use the law of sines to find $\angle B$.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$

$$\frac{\sin B}{40} = \frac{\sin 55°}{35}$$

$$\sin B = \frac{40 \sin 55°}{35} = \frac{40(0.8192)}{35} = 0.9362$$

$$\angle B = 69.4° \text{ or } 110.6°$$

If $\angle B = 69.4°$:

$$\angle C = 180° - 55° - 69.4° = 55.6°$$

$$c = \frac{a \sin C}{\sin A} = \frac{35 \sin 55.6°}{\sin 55°} = 35.3$$

If $\angle B = 110.6°$:

$$\angle C = 180° - 55° - 110.6° = 14.4°$$

$$c = \frac{a \sin C}{\sin A} = \frac{35 \sin 14.4°}{\sin 55°} = 10.6$$

$\therefore \angle B = 69.4°, \angle C = 55.6°$, and $c = 35.3$;
or $\angle B = 110.6°, \angle C = 14.4°$, and $c = 10.6$

Example 4 **(AAS case)** Solve $\Delta\, ABC$ if $a = 30$, $\angle A = 62°$, and $\angle B = 75°$.

Solution $\angle C = 180° - \angle A - \angle B = 180° - 62° - 75° = 43°$

Use the law of sines to find b and c.

$$b = \frac{a \sin B}{\sin A} = \frac{30 \sin 75°}{\sin 62°} = \frac{30(0.9659)}{0.8829} = 32.8$$

$$c = \frac{a \sin C}{\sin A} = \frac{30 \sin 43°}{\sin 62°} = \frac{30(0.6820)}{0.8829} = 23.2$$

$\therefore b = 32.8$, $c = 23.2$, and $\angle C = 43°$

Solve triangle *ABC*. If there are two solutions, find both. If there are no solutions, say so. Give lengths to three significant digits and angle measures to the nearest tenth of a degree. You may wish to use a calculator.

1. $a = 5, b = 7, \angle C = 80°$

2. $a = 2, \angle B = 40°, \angle C = 100°$

3. $a = 12, \angle A = 20°, \angle B = 60°$

4. $a = 5, b = 6, c = 7$

5. $a = 14, b = 9, \angle A = 62°$

6. $a = 8, b = 6, \angle B = 43°$

7. $b = 32, c = 30, \angle C = 78°$

8. $a = 50, c = 100, \angle A = 30°$

Study Guide, ALGEBRA AND TRIGONOMETRY, Structure and Method, Book 2
Copyright © by Houghton Mifflin Company. All rights reserved.

NAME ______________________________ DATE ______________

12–9 Areas of Triangles

Objective: To apply triangle area formulas.

Vocabulary

Triangle area formulas The area K of ΔABC is given by the following formulas:

$$K = \frac{1}{2}bc \sin A \qquad K = \frac{1}{2}ac \sin B \qquad K = \frac{1}{2}ab \sin C$$

$$K = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} \qquad K = \frac{1}{2}b^2 \frac{\sin A \sin C}{\sin B} \qquad K = \frac{1}{2}c^2 \frac{\sin A \sin B}{\sin C}$$

Hero's formula: $K = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$

Example 1 **(SAS case)** Find the area of ΔABC if $a = 24$, $b = 20$, and $\angle C = 76°$.

Solution Use the formula involving a, b, and $\angle C$.

$$K = \frac{1}{2}ab \sin C = \frac{1}{2}(24)(20) \sin 76°$$
$$= 240 \sin 76° = 240(0.9703) = 232.9$$

$\therefore$ to three significant digits, the area is 233 square units.

Example 2 **(ASA case)** Find the area of ΔABC if $a = 17.3$, $\angle B = 48°$, and $\angle C = 100°$.

Solution You can find $\angle A$ given $\angle B$ and $\angle C$: $\angle A = 180° - 48° - 100° = 32°$
Use the formula involving a, $\angle A$, $\angle B$, and $\angle C$.

$$K = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A} = \frac{1}{2}(17.3)^2 \frac{\sin 48° \sin 100°}{\sin 32°}$$
$$= 0.5(299.29) \frac{(0.7431)(0.9848)}{0.5299} = 206.7$$

$\therefore$ to three significant digits, the area is 207 square units.

Example 3 **(SSS case)** Find the area of ΔABC if $a = 30$, $b = 40$, and $c = 60$.

Solution Use Hero's formula: $K = \sqrt{s(s-a)(s-b)(s-c)}$

Find s: $s = \frac{1}{2}(a + b + c)$
$= 0.5(30 + 40 + 60) = 65$

Then: $s - a = 65 - 30 = 35$
$s - b = 65 - 40 = 25$
$s - c = 65 - 60 = 5$

Now substitute: $K = \sqrt{s(s-a)(s-b)(s-c)}$
$= \sqrt{65(35)(25)(5)} = 25\sqrt{455} = 533.3$

$\therefore$ to three significant digits, the area is 533 square units.

Copyright © by Houghton Mifflin Company. All rights reserved.

12–9 Areas of Triangles *(continued)*

Example 4 (SSA case) Find the area of $\triangle ABC$ if $a = 26$, $b = 34$, and $\angle A = 25°$.

Solution This is the ambiguous case discussed in Lesson 12–8. Since $b \sin A = 14$, it is less than both a and b. Thus two triangles are formed. Using the law of sines you will find that $\angle B$ is either 33.5° or 146.5°. Therefore:

$\angle C = 121.5°$ or $\angle C = 8.5°$

$$K = \frac{1}{2}ab \sin C \qquad K = \frac{1}{2}ab \sin C$$

$$= \frac{1}{2}(26)(34) \sin 121.5° \qquad = \frac{1}{2}(26)(34) \sin 8.5°$$

$$= 442(0.8526) \qquad = 442(0.1478)$$

$$= 376.8 \qquad = 65.32$$

$\therefore$ to three significant digits, the area is 377 square units or 65.3 square units.

Find the area of $\triangle ABC$ using the given data. Give area measure to three significant digits. You may wish to use a calculator.

1. $\angle A = 30°, \angle B = 65°, b = 25$
2. $a = 20, b = 15, \angle C = 48°$
3. $a = 12, c = 16, \angle B = 59°$
4. $\angle A = 80°, b = 16, c = 35$
5. $a = 14, b = 19, c = 23$
6. $a = 6, b = 7, c = 9$
7. $\angle B = 25°, \angle C = 110°, c = 14$
8. $b = 23, c = 14, \angle A = 145°$
9. $a = 12, b = 18, \angle C = 45°$
10. $a = 11, b = 12, \angle A = 60°$

Solve.

11. Find the area of a parallelogram that has a 65° angle and sides with lengths 8 and 15. (Hint: Divide the parallelogram into two equal triangles.)
12. Find the area of a rhombus that has perimeter 36 and an angle of 45°.

Mixed Review Exercises

Find the indicated part of $\triangle ABC$ to three significant digits or to the nearest tenth of a degree.

1. $a = 10, b = 8, \angle C = 65°, c = \underline{?}$
2. $\angle A = 110°, \angle B = 25°, b = 18, a = \underline{?}$
3. $a = 8, b = 6, c = 10, \angle C = \underline{?}$
4. $a = 20, \angle B = 80°, \angle C = 52°, b = \underline{?}$

Find the five other trigonometric functions of θ.

5. $\cos \theta = -\frac{1}{2}, 90° < \theta < 180°$
6. $\sin \theta = -\frac{3}{4}, 180° < \theta < 270°$
7. $\tan \theta = -1, 90° < \theta < 180°$
8. $\cos \theta = \frac{5}{13}, 270° < \theta < 360°$

Copyright © by Houghton Mifflin Company. All rights reserved.