NAME ______________________________ DATE ____________________

15 Statistics and Probability

15–1 *Presenting Statistical Data*

Objective: To display data using frequency distributions, histograms, and stem-and-leaf plots, and to compute measures of central tendency.

Vocabulary

Frequency distribution An organized listing of data that shows how many times each item or group of items occurs.

Histogram A method of displaying a frequency distribution using bars.

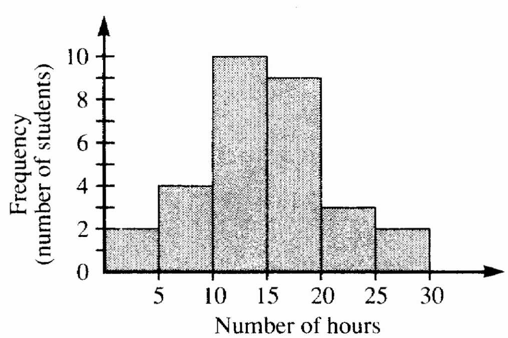
Stem-and-leaf plot A way of displaying data and including the data in the display.

Mode The number or numbers that occur most frequently in a set of data.

Median The middle number (when the count of the numbers is odd) or the arithmetic average of the two middle numbers (when the count is even) in a set of *ordered* data.

Mean Denoted by M, the mean is the arithmetic average of all the numbers in a set of data.

Example 1 Thirty high school students with after-school or weekend jobs were asked to give the number of hours per week they work. The results of this survey appear in the histogram below. Use the histogram to answer the questions.



Note: In this histogram, each boundary number on the horizontal axis belongs to the bar on the right. For example, the bar over the interval 10–15 includes those students who work from 10 to 15 hours, including 10 but *not* including 15.

a. How many students work at least 10 but fewer than 15 hours per week?

b. How many students work at least 20 hours per week?

Solution

a. The height of the bar over the interval 10–15 gives the number of students. ∴ 10 students work at least 10 but fewer than 15 hours per week.

b. Add the frequencies for the intervals 20–25 and 25–30: $3 + 2 = 5$ ∴ 5 students work at least 20 hours per week.

Use the histogram of Example 1 to answer the following questions.

1. How many students work at least 15 but fewer than 20 hours per week?
2. How many students work less than 10 hours per week?

15–1 Presenting Statistical Data *(continued)*

Example 2 Draw a stem-and-leaf plot for the following distribution of ages:
26, 30, 29, 41, 35, 26, 34, 29, 35, 30, 25, 42, 26, 34, 41, 35

Solution You first obtain the *stem* by using only the tens' digit of each number. These are written to the left of a vertical line. Then, for each age, you record the *leaf*, or units' digit, to the right of the stem. After making an unordered stem-and-leaf plot (as shown at the left below), you should reorder the elements of each row to complete the plot (as shown at the right below).

Stem	Leaf
2	6, 9, 6, 9, 5, 6
3	0, 5, 4, 5, 0, 4, 5
4	1, 2, 1

Stem	Leaf
2	5, 6, 6, 6, 9, 9
3	0, 0, 4, 4, 5, 5, 5
4	1, 1, 2

Example 3 Use the distribution of ages given in Example 2 to find (a) the mode, (b) the median, and (c) the mean.

Solution

a. Look for the numbers that occur most frequently. The ages 26 and 35 both occur three times, more often than any other age. Therefore, the distribution has two modes, 26 and 35.

b. Since there are 16 numbers in the distribution, there is no middle number. The median is the arithmetic average of the 8th and 9th ages counting from the top (or bottom) of the *ordered* list. Therefore, the median is $\frac{30 + 34}{2}$, or 32.

c. Since the mean is the arithmetic average of all 16 ages,

$$M = \frac{25 + 3(26) + 2(29) + 2(30) + 2(34) + 3(35) + 2(41) + 42}{16}$$
$$\approx 32.4$$

In Exercises 3–6, draw a stem-and-leaf plot for the given distribution.

3. 37, 41, 41, 42, 38, 38, 40, 41, 39, 39, 42, 37, 36

4. 61, 66, 92, 98, 72, 76, 79, 83, 92, 91, 85, 86, 76

5. 3, 7, 11, 5, 21, 27, 4, 8, 7, 21, 15, 17, 19, 21, 21, 8

6. 117, 115, 136, 142, 140, 141, 135, 135, 110, 153, 156, 147, 156, 115

7–10. Find (a) the mode, (b) the median, and (c) the mean for each distribution given in Exercises 3–6.

Mixed Review Exercises

Write in simplest form without negative exponents.

1. $\left(\frac{x^4y^{-2}}{x^{-3}y}\right)^{-3}\left(\frac{xy^{-1}}{x^2y}\right)$ **2.** $\frac{x^2 - 7x + 10}{x^3 - 125}$ **3.** $(x^{-1} + y^{-1})^{-1}(x + y)$ **4.** $9^{\sqrt{2}} \cdot 3^{\sqrt{2}}$

NAME ______________________ DATE ______________

15–2 Analyzing Statistical Data

Objective: To compute measures of dispersion and, together with measures of central tendency, to describe and compare distributions using these statistics.

Vocabulary

First quartile The median of the lower half of the data in a set of ordered data.

Third quartile The median of the upper half of the data in a set of ordered data.

Range The difference between the largest and smallest numbers in a set of data.

Box-and-whisker plot A diagram used to show the median, the first and third quartiles, and the range of a distribution.

Measures of central tendency The mean, the median, and the mode, each of which indicates a center of a distribution.

Variance The arithmetic average of the squares of the differences between the n numbers $x_1, x_2, \ldots, x_n$ and the mean M of a distribution:

$$\text{variance} = \frac{\text{sum of the squares of the deviations from the mean}}{\text{number of elements in the distribution}}$$

$$= \frac{(x_1 - M)^2 + (x_2 - M)^2 + \cdots + (x_n - M)^2}{n}$$

Standard deviation Denoted by the Greek letter σ (sigma), the standard deviation is the principal square root of the variance: $\sigma = \sqrt{\text{variance}}$.

Measures of dispersion The range, the first and third quartiles, the variance, and the standard deviation, each of which indicates how scattered a distribution is.

Example 1 For the distribution of children's weights (in pounds) shown in the stem-and-leaf plot at the right, find (a) the median, (b) the first quartile, (c) the third quartile, and (d) the range.

low end → 3	2, 4, 6
4	5, 7, 8, 9
5	1, 2, 2, 8, 9, 9
6	0, 5, 7, 8
7	0, 1, 2, 5 ← *high end*

Solution

a. With 21 weights in the distribution, the median is the 11th weight counting from either end: 58.

b. With 11 weights (including the median) in the lower half of the distribution, the first quartile is the 6th weight counting from the low end: 48.

c. With 11 weights (including the median) in the upper half of the distribution, the third quartile is the 6th weight counting from the high end: 67.

d. The difference between the largest and smallest weights is 75 − 32, or 43.

For each stem-and-leaf plot find (a) the median, (b) the first quartile, (c) the third quartile, and (d) the range.

1.

1	1, 5, 6
2	5, 5, 7, 7, 8
3	1, 2, 3, 4, 5, 6
4	0, 2, 3

2.

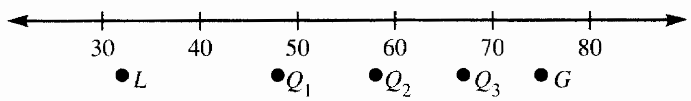
5	1, 8, 9
6	2, 2, 3, 4, 7, 9
7	3, 3, 4, 6, 8
8	6, 7

15–2 Analyzing Statistical Data *(continued)*

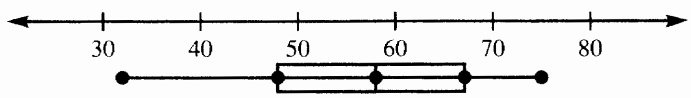
Example 2 Draw a box-and-whisker plot for the distribution in Example 1.

Solution

1. First, using dots below a number line, show the median (Q_2), the first and third quartiles (Q_1 and Q_3), and the least and greatest weights (L and G).



2. Next draw a narrow rectangular box with its shorter sides containing the two quartile dots. Then draw a line segment through the median dot parallel to the shorter sides. Finally, draw line segments, or "whiskers," from the first and third quartile dots to the least- and greatest-weight dots, respectively.



3–4. Draw a box-and-whisker plot for each of the distributions in Exercises 1 and 2 on page 241.

Example 3 The stem-and-leaf plot at the right shows the distribution of scores on an algebra test. Find (a) the mean, (b) the variance, and (c) the standard deviation for the distribution.

7	2, 3, 3, 3
8	0, 0, 0, 5, 5
9	3, 5, 5

Solution Use a table like the one shown below to organize your computations.

Score	Frequency	Score × Frequency	Deviation from the mean M	$(\text{Deviation})^2$	$(\text{Deviation})^2 \times$ Frequency
72	1	72	$72 - 82 = -10$	100	100
73	3	219	$73 - 82 = -9$	81	243
80	3	240	$80 - 82 = -2$	4	12
85	2	170	$85 - 82 = 3$	9	18
93	1	93	$93 - 82 = 11$	121	121
95	2	190	$95 - 82 = 13$	169	338
Sums	12	984			832

a. After completing the first three columns, you can compute the mean: $M = \frac{984}{12} = 82$

b. After completing the last three columns, you can compute the variance: variance $= \frac{832}{12} \approx 69.3$

c. Use the variance to compute the standard deviation: $\sigma \approx \sqrt{69.3} \approx 8.3$

For each distribution in Exercises 5–7 find (a) the mean, (b) the variance, and (c) the standard deviation to the nearest tenth.

5. 2, 3, 5, 5, 7, 8, 8, 10 **6.** 2, 3, 3, 5, 5, 5, 7, 10 **7.** 20, 22, 23, 23, 23, 24, 24, 25

NAME ______________________ DATE ______________

15-3 The Normal Distribution

Objective: To recognize and analyze normal distributions.

Vocabulary

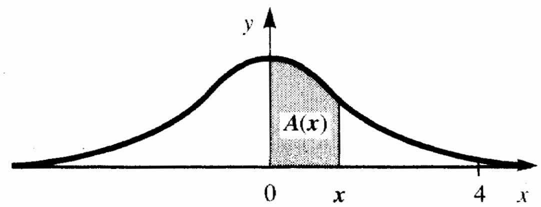
Normal distribution A distribution for which the bulk of the data is clustered about the mean, with the remainder tapering off away from the mean. A histogram of a normal distribution has a characteristic "bell-shaped" appearance.

Standard normal distribution A normal distribution with mean equal to zero and standard deviation equal to one.

Standard normal curve The bell-shaped curve that represents standard normal distribution. The standard normal curve has the following properties:

1. It is symmetric with respect to the y-axis.
2. As $|x|$ increases, the curve gets closer and closer to the x-axis.
3. The total area under the curve and above the x-axis is equal to 1.

Area under the Standard Normal Curve for $0 \le x \le 4.0$



x	Area, $A(x)$	x	Area, $A(x)$	x	Area, $A(x)$
0.0	0.0000	1.4	0.4192	2.8	0.4974
0.2	0.0793	1.6	0.4452	3.0	0.4987
0.4	0.1554	1.8	0.4641	3.2	0.4993
0.6	0.2257	2.0	0.4772	3.4	0.4997
0.8	0.2881	2.2	0.4861	3.6	0.4998
1.0	0.3413	2.4	0.4918	3.8	0.4999
1.2	0.3849	2.6	0.4953	4.0	0.5000

Standardized value The number x obtained from the number z, a given value from a normal distribution with mean M and standard deviation σ, using the formula:

$$x = \frac{z - M}{\sigma}$$

Example 1 In a standard normal distribution, what fraction of the data is between 2.0 and 2.4?

Solution The required fraction is equal to the area under the standard normal curve bounded by $x = 2.0$ and $x = 2.4$. This is the difference between the area from 0 to 2.4 and the area from 0 to 2.0. Therefore:

$$\begin{aligned} A(2.0 \le x \le 2.4) &= A(2.4) - A(2.0) \\ &= 0.4918 - 0.4772 \\ &= 0.0146 \end{aligned}$$

NAME ______________________ DATE ______________

15-3 The Normal Distribution *(continued)*

Example 2 In a standard normal distribution, what percent of the data is within one standard deviation of the mean?

Solution The mean is 0 and the standard deviation is 1. Since the standard normal curve is symmetric with respect to the y-axis, the area under the curve between -1 and 1 is twice the area between 0 and 1. Therefore:

$$\begin{aligned} A(-1 \le x \le 1) &= 2 \cdot A(1) \\ &= 2(0.3413) \\ &= 0.6826, \text{ or } 68.26\% \end{aligned}$$

In Exercises 1–3, use the table on page 243.

1. In a standard normal distribution, what fraction of the data is between the mean and 0.6?
2. In a standard normal distribution, what percent of the data is between -0.4 and 0.4?
3. In a standard normal distribution, what percent of the data is between two and three standard deviations below the mean?

Example 3 The heights of a certain group of high school girls have a normal distribution with a mean of 150 cm and a standard deviation of 5 cm. Find the percent of the group having a height greater than 158 cm.

Solution The standardized value corresponding to 158 is:

$$x = \frac{158 - M}{\sigma} = \frac{158 - 150}{5} = 1.6$$

The area under the standard normal curve to the right of $x = 1.6$ is the difference between the area to the right of $x = 0$ and the area between $x = 0$ and $x = 1.6$:

$$0.5 - 0.4452 = 0.0548$$

$\therefore$ about 5.5% of the group has a height greater than 158.

For Exercise 4, use the table on page 243.

4. The weights of a certain group of high school boys have a normal distribution with a mean of 146 lb and a standard deviation of 10 lb. Find the percent of the group having a weight (a) greater than 154 lb and (b) less than 132 lb.

Mixed Review Exercises

For the line containing the given points, find (a) the slope and (b) an equation in standard form.

1. $(2, -3), (5, 1)$
2. $(4, -3), (-3, 4)$
3. $(4, 7), (5, -2)$
4. $(-3, 4), (5, 4)$
5. $(-2, -3), (-4, -5)$
6. $(-1, -1), (-2, -2)$

NAME ______________________________ DATE ____________

15–4 Correlation

Objective: To draw a scatter plot, determine the correlation coefficient, and use the regression line for a set of ordered pairs of data.

Vocabulary

Scatter plot A graph of ordered pairs used to determine whether a mathematical relationship exists between two given sets of data.

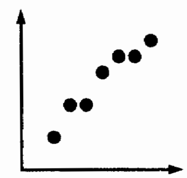
Correlation coefficient A measurement that shows how closely the points in a scatter plot cluster about a line. For a given set of ordered pairs (x, y), the correlation coefficient (denoted by r_{xy} or merely r) is given by the formula:

$$r = \frac{M_{xy} - M_x \cdot M_y}{\sigma_x \cdot \sigma_y}$$

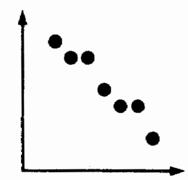
where M_x and σ_x are the mean and standard deviation of the x-values, M_y and σ_y are the mean and standard deviation of the y-values, and M_{xy} is the mean of the products of the ordered pairs.

The correlation coefficient is always a number between -1 and 1, inclusive. The more the ordered pairs cluster about a line with positive slope, the closer r is to 1. The more the ordered pairs cluster about a line with negative slope, the closer r is to -1. If the ordered pairs tend not to be collinear at all, the correlation coefficient is close to 0. (See the graphs at the right.)

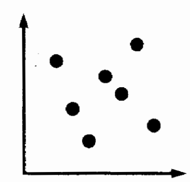
Regression line The line that best fits the known values of two variables when the correlation between the variables is high (that is, close to -1 or 1). For a set of ordered pairs (x, y), the regression line relating x and y contains the point (M_x, M_y) and has as its slope $r\left(\frac{\sigma_y}{\sigma_x}\right)$.



High positive correlation



High negative correlation



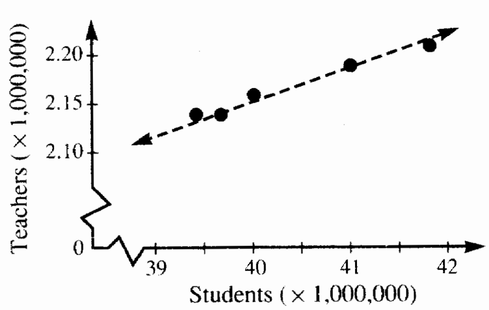
Almost no correlation

Example 1 **a.** Draw a scatter plot of the data given in the table at the right.

b. Describe the nature of the correlation between the number of students enrolled and the number of teachers employed in public elementary and secondary schools.

Public Elementary and Secondary Schools (five consecutive years)	
Number of Students (in millions)	Number of Teachers (in millions)
41.78	2.21
41.02	2.19
40.12	2.16
39.68	2.14
39.41	2.14

Solution **a.**



b. Since the ordered pairs tend to cluster about a line (shown dashed in the scatter plot) with a positive slope, there is a *high positive correlation* between the number of students and the number of teachers in public elementary and secondary schools.

NAME ______________________ DATE ______________

15–4 Correlation *(continued)*

Draw a scatter plot. Then describe the nature of the correlation.

1.

Manufacturers' Shipments of Televisions (five consecutive years)	
Color TVs (in millions)	Black and White TVs (in millions)
11.4	5.8
13.9	5.8
15.9	5.1
16.9	3.7
18.9	3.7

2.

Retail Sales of Passenger Cars in U.S. (five consecutive years)	
Domestic Cars (in millions)	Foreign Cars (in millions)
6.2	2.3
5.8	2.2
6.8	2.4
8.0	2.4
8.2	2.8

Example 2 For the table in Example 1, suppose x represents the number of students and y represents the number of teachers. Given that $M_{xy} \approx 87.616$, $M_x = 40.402$, $M_y = 2.168$, $\sigma_x \approx 0.879$, and $\sigma_y \approx 0.028$, determine (a) the correlation coefficient and (b) an equation of the regression line. Then (c) use the equation of the regression line to find the expected number of teachers when the number of students is 39 million.

Solution

a. The correlation coefficient of the ordered pairs is:

$$r = \frac{M_{xy} - M_x \cdot M_y}{\sigma_x \cdot \sigma_y} \approx \frac{87.616 - (40.402)(2.168)}{(0.879)(0.028)} \approx 0.99$$

b. The regression line contains the point $(M_x, M_y) = (40.402, 2.168)$.

The slope of the regression line is: $r\left(\frac{\sigma_y}{\sigma_x}\right) \approx 0.99\left(\frac{0.028}{0.879}\right) \approx 0.032$.

To obtain an equation of the regression line, use the point-slope form of the equation of a line:

$$y - 2.168 = 0.032(x - 40.402)$$
$$y = 0.032x + 0.875$$

c. Substituting 39 for x in the equation from part (b), you get a value of $0.032(39) + 0.875$, or 2.123, for y.

$\therefore$ the expected number of teachers is about 2.12 million.

3. For the table in Exercise 1, suppose x represents the number of color televisions and y represents the number of black and white televisions. Given that $M_{xy} \approx 72.06$, $M_x = 15.4$, $M_y = 4.82$, $\sigma_x \approx 2.57$, and $\sigma_y \approx 0.95$, determine (a) the correlation coefficient and (b) an equation of the regression line. Then (c) determine the expected number of black and white televisions when the number of color televisions is 20 million.

4. For the table in Exercise 2, suppose x represents the number of domestic cars and y represents the number of foreign cars. Given that $M_{xy} = 17.1$, $M_x = 7$, $M_y = 2.42$, $\sigma_x \approx 0.95$, and $\sigma_y \approx 0.20$, determine (a) the correlation coefficient and (b) an equation of the regression line. Then (c) determine the expected number of foreign cars when the number of domestic cars is 10 million.

NAME ______________________ DATE ______________

15–5 Fundamental Counting Principles

Objective: To apply fundamental counting principles.

Vocabulary

Mutually exclusive cases Two situations such that when one occurs, the other cannot. Example: A number can be less than 100 or greater than 100, but not both.

Fundamental counting principles There are two such principles, the first of which involves multiplication and the second, addition:

1. If one selection can be made in m ways, and for each of these a second selection can be made in n ways, then the number of ways the two selections can be made is $m \times n$.
2. If the possibilities being counted can be grouped into *mutually exclusive* cases, then the total number of possibilities is the sum of the number of possibilities in each case.

Example 1 How many *even* 2-digit positive integers less than 80 are there?

Solution A diagram such as [ ][ ] is useful to help analyze the problem.

Since the integers have 2 digits and are less than 80, there are *seven* possible selections for the tens' digit: 1, 2, 3, 4, 5, 6, and 7. Write 7 in the first box: [7][ ]

Since the integers are even, there are *five* possible selections for the units' digit: 0, 2, 4, 6, and 8. Write 5 in the second box: [7] [5]

∴ by the first fundamental counting principle there are $7 \cdot 5$, or 35, even 2-digit positive integers less than 80.

Example 2 How many positive integers less than 100 can be written using the digits 1, 3, 5, 7, and 9?

Solution Consider two mutually exclusive cases: (1) the 1-digit integers and (2) the 2-digit integers.

	Tens	Units	Number of integers	
1-digit integers	[–]	[5]	5	
2-digit integers	[5]	[5]	$5 \cdot 5 = 25$	by the first fundamental counting principle

∴ by the second fundamental counting principle there are $5 + 25$, or 30, positive integers less than 100 that can be written using the digits 1, 3, 5, 7, and 9.

NAME ______________________________ DATE ______________

15-5 *Fundamental Counting Principles* *(continued)*

Example 3 In how many ways can a 5-question true-false section of a test be answered:
a. if every question must be answered?
b. if it is all right to leave questions unanswered?

Solution **a.** Each of the 5 questions has *two* possible answers: true or false.
Use a diagram where each box represents a question: [2] [2] [2] [2] [2]

$\therefore$ there are 2^5, or 32, possible ways to answer the true-false section of the test if every question must be answered.

b. Each of the 5 questions has *three* possible answers: true, false, or blank.
Use a diagram where each box represents a question: [3] [3] [3] [3] [3]

$\therefore$ there are 3^5, or 243, possible ways to answer the true-false section of the test if it is all right to leave questions unanswered.

1. How many odd 2-digit positive integers less than 50 are there?

2. How many even 2-digit positive integers greater than 60 are there?

3. How many positive integers less than 1000 can be written using the digits 3, 4, 5, and 6?

4. How many even 3-digit positive integers can be written using the digits 1, 3, 4, 5, and 6?

5. In how many different ways can an 8-question true-false test be answered:
a. if every question must be answered?
b. if it is all right to leave questions unanswered?

6. A bookshelf contains 6 different algebra books, 5 different geometry books, and 3 different trigonometry books. In how many ways can you select:
a. any one of the math books?
b. one of each of the three types of math books?

7. In how many ways can you select 4 cards, one after another, from a 52-card deck:
a. if the cards are returned to the deck after being selected?
b. if the cards are *not* returned to the deck after being selected?

Mixed Review Exercises

Evaluate if $x = -3$ and $y = 4$.

1. $\frac{y^3}{x^4}$ **2.** $|y - x|$ **3.** y^x **4.** $\sqrt{xy}$

Use the given root to solve each equation completely.

5. $x^3 - x^2 - x - 2 = 0$; 2 **6.** $x^3 - 3x^2 + 4x - 12 = 0$; $2i$

7. $2x^3 - x^2 + 18x - 9 = 0$; $3i$ **8.** $2x^3 + 3x^2 - 4x - 4 = 0$; -2

NAME ______________________________ DATE ______________

15–6 Permutations

Objective: To find the number of permutations of the elements of a set.

Vocabulary

Factorial of a positive integer The product of the given integer and all positive integers less than the given integer. Example: The factorial of 5 (written 5!) is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, or 120. (*Note*: 0! is defined to be 1.)

Permutation An arrangement of the elements of a set *in a definite order*. Example: Six permutations of the elements of $\{m, n, o\}$ are possible: *mno, mon, nmo, nom, omn,* and *onm*. Permutations have the following properties:

1. The number of permutations of n objects is $n!$.
2. The number of permutations of n objects taken r at a time, denoted ${}_nP_r$, is given by the formula ${}_nP_r = \dfrac{n!}{(n - r)!}$.
3. If a set of n elements has n_1 elements of one kind alike, n_2 of another kind alike, and so on, then the number of permutations, P, of the n elements taken n at a time is given by $P = \dfrac{n!}{n_1!n_2! \cdots}$.

Symbols

! (factorial) ${}_nP_r$ (the number of permutations of n objects taken r at a time)

Example 1 Evaluate: **a.** $4!\,6!$ **b.** $\dfrac{6!}{(6-2)!}$

Solution **a.** $4!\,6! = (4 \cdot 3 \cdot 2 \cdot 1) \cdot (6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 17{,}280$

b. $\dfrac{6!}{(6-2)!} = \dfrac{6!}{4!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 30$

Evaluate.

1. $3!\,4!$ **2.** $2(6!)$ **3.** $\dfrac{7!}{(7-3)!}$ **4.** $\dfrac{7!}{2!\,5!}$

Example 2 Find the number of permutations of 5 objects taken:
a. 5 at a time. **b.** 3 at a time.

Solution **a.** Use the second property of permutations:

$${}_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120$$

(Notice that this result agrees with the first property of permutations.)

b. ${}_5P_3 = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$

Find ${}_nP_r$ for the given values of n and r.

5. $n = 8, r = 8$ **6.** $n = 6, r = 2$ **7.** $n = 9, r = 1$ **8.** $n = 10, r = 5$

NAME ______________________________ DATE ______________

15-6 Permutations *(continued)*

Example 3 In how many ways can 6 potted plants be arranged on a windowsill if there is room: **a.** for all 6 pots? **b.** for only 5 pots?

Solution **a.** You are looking for the number of permutations of 6 objects, which is just 6!, or 720.

∴ there are 720 ways to arrange 6 potted plants on a windowsill.

b. You are looking for the number of permutations of 6 objects taken 5 at a time:

$${}_6P_5 = \frac{6!}{(6-5)!} = \frac{6!}{1!} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$$

∴ there are 720 ways to arrange any 5 of 6 potted plants on a windowsill.

Example 4 In how many ways can the letters in the word BASE be arranged using: **a.** all 4 letters? **b.** only 2 letters at a time?

Solution **a.** $4! = 24$

∴ there are 24 ways to arrange the letters in the word BASE.

b. $${}_4P_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \cdot 3 = 12$$

∴ there are 12 ways to arrange the letters in the word BASE using only 2 letters at a time.

9. In how many ways can 8 different cars parallel park along one side of a street if there is room: **a.** for all 8 cars? **b.** for only 5 cars?

10. In how many ways can 10 different books be arranged on a shelf if there is room: **a.** for all 10 books? **b.** for only 7 books?

11. In how many ways can the letters of the word UMPIRE be arranged using: **a.** all 6 letters? **b.** only 4 letters?

12. In how many ways can the letters of the word PITCHER be arranged using: **a.** all 7 letters? **b.** only 3 letters?

Example 5 In how many ways can the letters in the word BASEBALL be arranged?

Solution There are 8 letters, of which 2 are B's, 2 are A's, and 2 are L's. Use the third property of permutations:

$$\frac{8!}{2!\,2!\,2!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8} = 5040$$

∴ there are 5040 ways to arrange the letters in the word BASEBALL.

Find the number of ways the letters of each word can be arranged.

13. SOCCER **14.** RUNNING **15.** FOOTBALL **16.** VOLLEYBALL

NAME ______________________ DATE ______________

15–7 Combinations

Objective: To find the combinations of a set of elements.

Vocabulary

Subset Set B is a subset of set A if each member of B is also a member of A. Example: If $C = \{1, 2, 3\}$ and $D = \{1, 2\}$, then D is a subset of C, but C is not a subset of D. (*Note*: The empty set, $\emptyset$, is a subset of every set.)

Combination An r-element subset of a set of n elements is called a combination of n elements taken r at a time. The number of combinations of n elements taken r at a time, denoted ${}_nC_r$, is given by:

$$ {}_nC_r = \frac{{}_nP_r}{{}_rP_r} = \frac{n!}{r!\,(n-r)!} $$

Symbol

${}_nC_r$ (the combination of n elements taken r at a time)

CAUTION Recall that the order of the elements matters for permutations: MN and NM, for example, are different permutations. The order of the elements is *not* important for combinations, however: {M, N} and {N, M} are the same combination.

Example 1 For the 3-digit set $\{2, 4, 6\}$, find:
a. all the subsets. **b.** the 2-digit subsets.

Solution **a.** $\{2, 4, 6\}, \{2, 4\}, \{2, 6\}, \{4, 6\}, \{2\}, \{4\}, \{6\}, \emptyset$
b. $\{2, 4\}, \{2, 6\}, \{4, 6\}$

1. For the two-digit set $\{1, 3\}$, find:
   a. all the subsets. **b.** the subsets containing at least one digit.
2. For the four-letter set {A, B, C, D}, find:
   a. the 3-letter subsets. **b.** the subsets containing fewer than three letters.

Example 2 Evaluate: **a.** ${}_4C_3$ **b.** ${}_{10}C_6$

Solution **a.** ${}_4C_3 = \frac{4!}{3!\,1!} = \frac{4 \cdot 3!}{3! \cdot 1} = \frac{4}{1} = 4$

b. ${}_{10}C_6 = \frac{10!}{6!\,4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$

Evaluate.

3. ${}_6C_2$ **4.** ${}_5C_1$ **5.** ${}_8C_5$ **6.** ${}_7C_3$

7. ${}_{11}C_9$ **8.** ${}_{20}C_4$ **9.** ${}_{52}C_3$ **10.** ${}_{96}C_2$

NAME ______________________________ DATE ______________

15–7 Combinations *(continued)*

Example 3 Find the number of combinations of the letters in the word GRADE, taking them (a) 5 at a time and (b) 2 at a time. List each combination.

Solution

a. ${}_5C_5 = \frac{5!}{5!\,0!} = 1$ {G, R, A, D, E}

b. ${}_5C_2 = \frac{5!}{2!\,3!} = 10$ {G, R}, {G, A}, {G, D}, {G, E}, {R, A} {R, D}, {R, E}, {A, D}, {A, E}, {D, E}

11. How many combinations can be formed from the letters in NUMBER, taking them:
a. 5 at a time? **b.** 3 at a time? **c.** 2 at a time?

12. How many combinations can be formed from the letters in EQUATION, taking them:
a. 7 at a time? **b.** 5 at a time? **c.** 2 at a time?

Example 4 In how many ways can a basketball team having 5 players be chosen from 6 seniors and 4 juniors:
a. if all are equally eligible? **b.** if the team must have 3 seniors and 2 juniors?

Solution

a. There are 10 people eligible for the team: ${}_{10}C_5 = \frac{10!}{5!\,5!} = 252$

b. There are ${}_6C_3$ ways to choose seniors and ${}_4C_2$ ways to choose juniors. Use the first fundamental counting principle to find the number of ways to select the team: ${}_6C_3 \cdot {}_4C_2 = \frac{6!}{3!\,3!} \cdot \frac{4!}{2!\,2!} = 20 \cdot 6 = 120$

13. A sample of 4 light bulbs taken from a group of 50 light bulbs is to be inspected. How many different samples could be selected?

14. Seven boys and 8 girls want to participate in a school play. In how many ways can a cast be selected if it must have 4 boys and 5 girls?

Mixed Review Exercises

Find the value of each function if $x = 3$.

1. $f(x) = \frac{2x + 5}{3x - 2}$ **2.** $g(x) = |5 - 2x|$ **3.** $h(x) = (3 - x)^2$

4. $F(x) = -5$ **5.** $G(x) = -\frac{2}{3}x + 4$ **6.** $H(x) = x^3 - 27$

Simplify.

7. $\sqrt{48} - \sqrt{8} + \sqrt{2}$ **8.** $\log_5 100 - 2 \log_5 2$ **9.** $\frac{1}{x + 3} + \frac{2}{x^2 + x - 6}$

10. $(5 - 2i)(3 + i)$ **11.** $\frac{4x^2 + 6x + 9}{8x^3 - 27}$ **12.** $\left(\frac{\sqrt{11} + 1}{4}\right)\left(\frac{\sqrt{11} - 1}{4}\right)$

NAME ______________________ DATE ______________

15–8 Sample Spaces and Events

Objective: To specify sample spaces and events for random experiments.

Vocabulary

Random experiment An experiment in which you do not necessarily get the same outcome when you repeat it under the same conditions. Example: Tossing a coin is a random experiment.

Sample space The set of all possible outcomes of a random experiment. Example: If H represents heads and T tails, then $\{H, T\}$ is the sample space for a coin tossing experiment.

Event Any subset of possible outcomes for an experiment.

Simple event An event that represents a single element of a random experiment's sample space. Example: $\{H\}$ and $\{T\}$ are each simple events for a coin tossing experiment.

Example 1 Each letter of the word RANDOM is written on a separate card. The cards are then shuffled, and one card is drawn.

a. Specify the sample space for the experiment.
b. Specify the event that the letter on the card that is drawn is a vowel.
c. Specify the event that the letter on the card that is drawn is neither M nor N.

Solution

a. {R, A, N, D, O, M}
b. {A, O}
c. {R, A, D, O}

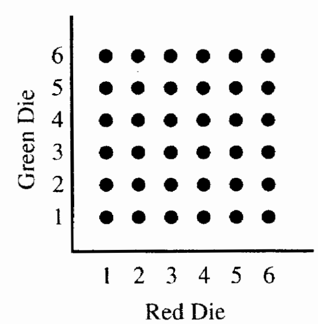
Example 2 Suppose you roll two dice, one red and one green.

a. Specify the sample space for the experiment.
b. Specify the event that both dice turn up the same number.

Solution

a. The outcome can be represented by the ordered pair (r, g) where r is the number on the red die and g is the number on the green die.

{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}



Another way of specifying this sample space is shown at the right. The dot in the upper left corner, for example, represents the simple event "red die shows 1 and green die shows 6."

b. {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

15–8 *Sample Spaces and Events* *(continued)*

Exercises 1–3 refer to an experiment in which each letter of the word SAMPLE is written on a separate card. The cards are then shuffled, and one card is drawn.

1. Specify the sample space for the experiment.
2. Specify the event that the letter on the card that is drawn is a consonant.
3. Specify the event that the letter on the card that is drawn comes before N in the alphabet.

For Exercises 4–9, refer to the two-dice experiment described in Example 2. Specify each event.

4. The sum of the numbers showing on the two dice is 4.
5. The product of the numbers showing on the two dice is 6.
6. The sum of the numbers showing on the two dice is greater than 9.
7. The product of the numbers showing on the two dice is greater than 20.
8. The number on the red die is one more than the number on the green die.
9. The number on the green die is 5, and the number on the red die is greater than 3.

Example 3 One box contains a red ball and a blue ball. A second box contains a red ball, a white ball, and a green ball. One ball is drawn from each box. For the experiment specify:

a. the sample space.
b. the event that one ball is green.
c. the event that at least one ball is red.
d. the event that exactly one ball is red.

Solution

a. An outcome of the experiment can be represented by the ordered pair (b_1, b_2) where b_1 is the ball drawn from the first box and b_2 is the ball drawn from the second box. If R represents a red ball, B a blue ball, W a white ball, and G a green ball, then the sample space for the experiment is $\{(R, R), (R, W), (R, G), (B, R), (B, W), (B, G)\}$.

b. $\{(R, G), (B, G)\}$

c. $\{(R, R), (R, W), (R, G), (B, R)\}$

d. $\{(R, W), (R, G), (B, R)\}$

10. Two bags contain marbles. The first bag contains a white, an orange, and a yellow marble. The second bag contains a white and an orange marble. One marble is drawn from each bag. For the experiment specify:

    a. the sample space.
    b. the event that one marble is yellow.
    c. the event that at most one marble is white.
    d. the event that neither marble is orange.

NAME ______________________ DATE ______________

15–9 Probability

Objective: To find the probability that an event will occur.

Vocabulary

Probability A number (between 0 and 1, inclusive) that expresses the likelihood of an event occurring. Example: The probability of obtaining a head on the toss of a coin is $\frac{1}{2}$; this is written $P(H) = \frac{1}{2}$. Some properties of probability are:

1. If $\{a_1, a_2, a_3, \ldots, a_n\}$ is a sample space containing n equally likely outcomes, then the probability of each simple event is $\frac{1}{n}$:

$$P(a_1) = P(a_2) = P(a_3) = \cdots = P(a_n) = \frac{1}{n}$$

2. If the sample space for an experiment consists of n equally likely outcomes, and if k of them are in event E, then:

$$P(E) = \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{k \text{ addends}} = \frac{k}{n}$$

3. If event E contains all elements of the sample space, then $P(E) = \frac{n}{n} = 1$. That is, the event is certain to occur.

4. If event E contains no elements of the sample space, then $P(E) = \frac{0}{n} = 0$. That is, the event is certain *not* to occur.

Symbol

$P(E)$ (the probability of event E)

Example 1 A die is rolled. Find the probability of each event.

a. Event A: The number showing is even.

b. Event B: The number showing is 5 or 6.

Solution The sample space is $\{1, 2, 3, 4, 5, 6\}$. Since the six outcomes are equally likely, the probability of each simple event is $\frac{1}{6}$.

a. Event $A = \{2, 4, 6\}$

$P(A) = \frac{3}{6} = \frac{1}{2}$

b. Event $B = \{5, 6\}$

$P(B) = \frac{2}{6} = \frac{1}{3}$

1. A number wheel is divided into 12 congruent sectors, numbered 1–12. You spin the wheel and record the number. Find the probability of each event.

a. It is a 5.

b. It is a 2-digit number.

c. It is a factor of 12.

d. It is an odd number.

e. It is a negative number.

f. It is a factor of 27,720.

15–9 Probability *(continued)*

2. A letter is selected at random from those in the word RECTANGLE. Find the probability of each event.
   a. It is a vowel.
   b. It is a consonant.
   c. It is between D and M in the alphabet.
   d. It is either C or G.

3. One marble is drawn at random from a bag containing 2 red, 4 yellow, and 6 blue marbles. Find the probability of each event.
   a. It is yellow.
   b. It is blue.
   c. It is not yellow.
   d. It is red or blue.

4. One card is drawn at random from a 52-card deck. Find the probability of each event.
   a. It is a king.
   b. It is a heart.
   c. It is red.
   d. It is the ace of spades.
   e. It is a red jack.
   f. It is red or black.

Example 2 Two coins are tossed. Find the probability of each event.

a. Event A: Both come up heads.
b. Event B: Each coin comes up either heads or tails.
c. Event C: Neither coin comes up heads.
d. Event D: One coin comes up heads and the other coin comes up tails.

Solution The sample space is $\{(H, T), (H, H), (T, H), (T, T)\}$.

a. Event $A = \{(H, H)\}$

$P(A) = \frac{1}{4}$

b. Event $B = \{(H, T), (H, H), (T, H), (T, T)\}$

$P(B) = \frac{4}{4} = 1$

c. Event $C = \{(T, T)\}$

$P(C) = \frac{1}{4}$

d. Event $D = \{(H, T), (T, H)\}$

$P(D) = \frac{2}{4} = \frac{1}{2}$

5. Two dice are rolled. Find the probability of each event.
   a. The sum of the numbers showing on the dice is 7.
   b. The sum of the numbers showing on the dice is less than 6.
   c. Exactly one die shows 3.
   d. At least one die shows 3.

Mixed Review Exercises

Solve each inequality and graph the solution set.

1. $|x + 3| > 5$
2. $y^2 + y \le 6$
3. $3 - 2w > -1$
4. $-2 \le 3m + 4 \le 4$
5. $8u^2 > 32$
6. $|1 - 2k| < 3$
7. How many permutations are there of all the letters in AXIOM?
8. In how many ways can a committee of 4 be selected from a club having 16 members?

NAME ______________________ DATE ______________

15–10 Mutually Exclusive and Independent Events

Objective: To identify mutually exclusive and independent events and find the probability of such events.

Vocabulary

Intersection of sets The intersection of sets A and B is the set whose members are elements of both A and B. Example: If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then the intersection of A and B (written $A \cap B$) is $\{3\}$.

Disjoint sets Sets that have no members in common. (*Note*: A and B are disjoint sets if and only if $A \cap B = \emptyset$.)

Union of sets The union of sets A and B is the set whose members are the elements belonging to either A or B (or both). Example: If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then the union of A and B (written $A \cup B$) is $\{1, 2, 3, 4\}$.

Mutually exclusive events Two events that have no simple events in common. Example: Getting two tails and getting at least one head are mutually exclusive events when two coins are tossed.

Independent events Two events such that the occurrence of one has no effect on the probability of the other.

Complement of an event If event A is a subset of sample space S, the complement of A (written $\overline{A}$) consists of the elements of S that are *not* members of A.

Probability Here are some additional properties of probability:

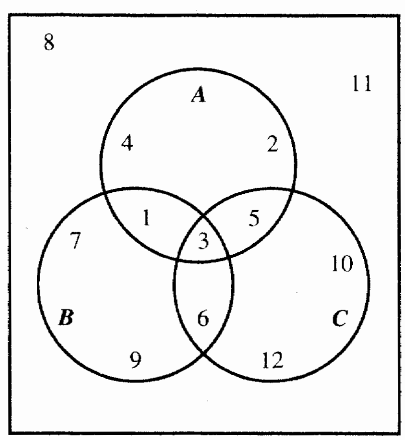
1. For any sample space S, $P(S) = 1$ and $P(\emptyset) = 0$.
2. For any two events A and B in a sample space, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
3. If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$.
4. Two events A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.
5. For any event A, $P(A) + P(\overline{A}) = 1$, or $P(\overline{A}) = 1 - P(A)$.

Symbols

$\cap$ (intersection) $\cup$ (union) $\overline{A}$ (the complement of event A)

Example 1 Specify each of the following sets by listing its elements. Use the *Venn diagram* (a diagram used to picture sets and set relationships) shown at the right.

a. $A \cap B$ **b.** $B \cap C$
c. $A \cup B$ **d.** $B \cup \overline{C}$
e. $(A \cup B) \cap C$ **f.** $A \cap \overline{C}$



Solution **a.** $\{1, 3\}$ **b.** $\{3, 6\}$
c. $\{1, 2, 3, 4, 5, 6, 7, 9\}$
d. $\{1, 3, 5, 6, 7, 9, 10, 12\}$
e. $\{3, 5, 6\}$ **f.** $\{1, 2, 4\}$

NAME ______________________ DATE ______________

15–10 Mutually Exclusive and Independent Events *(continued)*

1. Use the Venn diagram given in Example 1 to specify each of the following sets by listing its elements.
a. $A \cap C$ **b.** $A \cup C$ **c.** $(A \cap B) \cup C$ **d.** $\bar{A} \cap B$

Example 2 A letter is selected at random from those in the word FOR. A second letter is selected at random from those in the word SOME. Let A be the event that each letter is an O, and let B be the event that neither letter is an O.

a. How many elements are in the sample space of the experiment?
b. Specify the simple events in A, B, $A \cap B$, and $A \cup B$.
c. Find the probability of A, B, $A \cap B$, and $A \cup B$.
d. Confirm that the second property of probability given on page 257 holds.
e. Are A and B mutually exclusive? Are they independent?

Solution

a. The word FOR contains 3 letters, and the word SOME contains 4. By the first fundamental counting principle, there are $3 \cdot 4$, or 12, ways of selecting a letter from FOR and then a letter from SOME.

b. Let the ordered pair (l_1, l_2) represent the selection of the letter l_1 from FOR followed by the selection of the letter l_2 from SOME.

$A = \{(O, O)\}$
$B = \{(F, S), (F, M), (F, E), (R, S), (R, M), (R, E)\}$
$A \cap B = \emptyset$
$A \cup B = \{(F, S), (F, M), (F, E), (O, O), (R, S), (R, M), (R, E)\}$

c. $P(A) = \frac{1}{12}$ $P(B) = \frac{6}{12} = \frac{1}{2}$

$P(A \cap B) = \frac{0}{12} = 0$ $P(A \cup B) = \frac{7}{12}$

d. $P(A) + P(B) - P(A \cap B) = \frac{1}{12} + \frac{6}{12} - 0 = \frac{7}{12} = P(A \cup B)$

e. Since $A \cap B$ is empty, A and B are mutually exclusive. Since $P(A) \cdot P(B) = \frac{1}{12} \cdot \frac{1}{2} = \frac{1}{24} \neq P(A \cap B)$, A and B are not independent.

2. A bag contains 1 red, 1 white, and 1 blue marble. A marble is drawn at random from the bag and then put back before a second marble is drawn. Let A be the event that the first marble drawn is red, and let B be the event that the second marble drawn is not red.
a. How many elements are in the sample space of the experiment?
b. Specify the simple events in A, B, $A \cap B$, and $A \cup B$.
c. Find the probability of A, B, $A \cap B$, and $A \cup B$.
d. Confirm that the second property of probability given on page 257 holds.
e. Are A and B mutually exclusive? Are they independent?

3. Repeat Exercise 2, this time assuming that the first marble is *not* put back before the second is drawn.