Problem 5, p. 82 — If a ball is thrown into the air with a velocity of $40 \frac{ft}{s}$, its height in feet t seconds later is given by the formula

$$y = 40t - 16t^2. \quad \text{(from physics)}$$

(a) Find the average velocity for the time period beginning when $t = 2$ sec and lasting

(i) 0.5 seconds.

Soln. Let $\bar{v}$ denote the average velocity.

$\bar{v} = \frac{\Delta y}{\Delta t}$, where Δy denotes the change in height and Δt is the elapsed time. At time $t = 2$ seconds,

$$y = 40(2) - 16(2^2) = 16 \text{ ft}.$$

At $t = 2.5$ s,

$$y = 40(2.5) - 16(2.5)^2 = 0 \text{ ft}.$$

Therefore,

$$\bar{v} = \frac{\Delta y}{\Delta t} = \frac{0 \text{ ft} - 16 \text{ ft}}{2.5 s - 2 s} = -32 \frac{ft}{s}$$

(ii) 0.1 seconds.

$$\bar{v} = \frac{40(2.1) - 16(2.1)^2 - 16}{2.1 - 2} = -25.6 \frac{ft}{s}$$

(iii) 0.05 seconds.

$$\bar{v} = \frac{40(2.05) - 16(2.05)^2 - 16}{2.05 - 2} = -24.8 \frac{ft}{s}$$

(iv) 0.01 seconds.

$$\bar{v} = \frac{40(2.01) - 16(2.01)^2 - 16}{2.01 - 2} = -24.16 \frac{ft}{s}$$

Let's also find the average velocity from $t = 2$ seconds to $t = 2.0001$ seconds:

$$\bar{v} = \frac{40(2.0001) - 16(2.0001)^2 - 16}{2.0001 - 2} = \frac{-.0024\,0016}{.0001} = -24.0016 \ \frac{ft}{s}$$

(b) Estimate the instantaneous velocity when $t = 2$ seconds.

Soln. Note that as $\Delta t \to 0$ (i.e., as Δt approaches 0), Δy also approaches 0. But it is incorrect (and nonsense) to say $\bar{v} \to \frac{0}{0}$. What we see for $\bar{v}$ is the sequence of values:

$$-32, -25.6, -24.8, -24.16, -24.0016, \ldots$$

It appears that $\bar{v} \to -24 \frac{ft}{s}$ as $\Delta t \to 0$. Thus we estimate that the instantaneous velocity is $-24 \frac{ft}{s}$.

We finished the problem, but let's rework it using a more efficient process as follows:

Let Δt denote the elapsed time (let's not give it a value yet). Then the average velocity between the times 2 and $2 + \Delta t$ sec is

$$\bar{v} = \frac{[40(2+\Delta t) - 16(2+\Delta t)^2] - 16}{(2+\Delta t) - 2} = \frac{80 + 40\Delta t - 16(4 + 4\Delta t + (\Delta t)^2) - 16}{\Delta t}$$

$$= \frac{80 + 40\Delta t - 64 - 64\Delta t - 16(\Delta t)^2 - 16}{\Delta t} = \frac{-24\Delta t - 16(\Delta t)^2}{\Delta t}$$

$$= \frac{\Delta t(-24 - 16\Delta t)}{\Delta t} = -24 - 16\Delta t.$$

Thus, the average velocity between the times 2 and $2+\Delta t$ seconds is

$$\boxed{\bar{v} = -24 - 16\Delta t}$$

So,

(i) From 2 s to 2.5 sec,

$$\bar{v} = -24 - 16(0.5) = -24 - 8 = -32 \frac{ft}{s}$$

(ii) From 2 s to 2.1 s,

$$\bar{v} = -24 - 16(0.1) = -24 - 1.6 = -25.6 \frac{ft}{s}$$

(iii) From 2 s to 0.05 sec,

$$\bar{v} = -24 - 16(0.05) = -24.8 \frac{ft}{s}$$

(iv) From 2 s to 0.01 seconds,

$$\bar{v} = -24 - 16\Delta t = -24 - 16(.01) = -24.16 \frac{ft}{s}$$

Finally, observe that as $\Delta t \to 0$,

$$-24 - 16\Delta t \to -24 - 16(0) = -24.$$

We express this symbolically by writing

$$\lim_{\Delta t \to 0} (-24 - 16\Delta t) = -24.$$

We define this to be the instantaneous velocity of the ball at $t = 2$ seconds.