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8 Variation and Polynomial Equations

8–1 Direct Variation and Proportion

Objective: To solve problems involving direct variation.

Vocabulary

Direct variation A linear function defined by an equation of the form $y = mx$ ($m \neq 0$). The constant m in the equation is called the *constant of variation* or *constant of proportionality*. We say that *y varies directly as x* because if x increases, y also increases, and if x decreases, y also decreases.

Proportion An equality of ratios. A proportion can be written in the form

$\frac{y_1}{x_1} = \frac{y_2}{x_2}$, or $y_1 : x_1 = y_2 : x_2$, where the ordered pairs (x_1, y_1) and (x_2, y_2) are solutions of a direct variation, and $\frac{y_1}{x_1} = \frac{y_2}{x_2} = m$ (x_1 and $x_2 \neq 0$).

In a direct variation, y is often said to be *directly proportional* to x.

Means and extremes In the proportion $y_1 : x_1 = y_2 : x_2$, the means are the numbers x_1 and y_2, and the extremes are y_1 and x_2. In a proportion, *the product of the extremes equals the product of the means*, or $y_1 x_2 = x_1 y_2$.

Example 1 If y varies directly as x, and $y = 12$ when $x = 20$, find y when $x = 50$.

Solution First find m and write an equation of the direct variation.

$y = mx$	Start with the general equation.
$12 = m(20)$	Substitute $y = 12$ and $x = 20$.
$m = \frac{12}{20} = \frac{3}{5}$	Solve for m.

$\therefore\ y = \frac{3}{5}x$ is an equation of the direct variation.

You can find y when $x = 50$ by substituting 50 for x in this equation.

$$y = \frac{3}{5}(50) = 30$$

Example 2 If a is directly proportional to $b + 3$, and $a = 6$ when $b = 15$, find b when $a = 7$.

Solution Since a is directly proportional to $b + 3$, you can write a proportion.

$\frac{a_1}{b_1 + 3} = \frac{a_2}{b_2 + 3}$	Set up a proportion.
$\frac{6}{15 + 3} = \frac{7}{b_2 + 3}$	Substitute values for variables.
$6(b_2 + 3) = 18(7)$	Multiply the extremes, and the means.
$b_2 + 3 = 126 \div 6 = 21$	Solve for b_2.
$b_2 = 18$	

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8–1 Direct Variation and Proportion *(continued)*

CAUTION It doesn't matter whether you solve direct variation problems by finding the equation first or by using the proportion method. However, if you decide to use the proportion method, be sure that the values in the numerators represent the same variable.

Solve.

1. If y varies directly as x, and $y = 6$ when $x = 4$, find y when $x = 12$.
2. If a is directly proportional to b, and $a = 25$ when $b = 35$, find b when $a = 40$.
3. If w varies directly as z, and $w = 4.5$ when $z = 3$, find z when $w = 1.5$.
4. If p is directly proportional to q^3, and $p = 3$ when $q = 2$, find p when $q = 4$.
5. If r varies directly as $s + 1$, and $r = 4$ when $s = 5$, find r when $s = 8$.
6. If a varies directly as $3b + 2$, and $a = 10$ when $b = 6$, find b when $a = 7$.

Example 3 If a car travels 70 km in 2 hours, how far can it travel in 4.5 hours, traveling at the same rate of speed?

Solution Let d be the required distance in kilometers.

Since the ratio $\frac{distance}{time} = rate$ is constant, a proportion can be written.

$$\frac{d_1}{t_1} = \frac{d_2}{t_2} \longrightarrow \frac{70}{2} = \frac{d}{4.5}$$

$$2d = 70(4.5)$$
$$d = 157.5$$

$\therefore$ the distance the car will travel is 157.5 km.

Solve.

7. If the sales tax on a \$38 purchase is \$2.85, what will the tax be on an \$84 purchase?
8. A survey showed that 52 out of 234 people questioned preferred hot cereal to cold cereal. In a school population of 1800, how many people are likely to prefer hot cereal?
9. A real estate agent received a commission of \$2232 on a piece of land that sold for \$124,000. At this rate, what commission will the agent receive for a piece of land that sold for \$160,000?

Mixed Review Exercises

Solve each equation over the real numbers.

1. $4x^2 - 7x + 2 = 0$
2. $\frac{a - 1}{a + 5} = 1 - \frac{3}{a}$
3. $\frac{3y^2}{8} + \frac{y}{4} = 1$
4. $|2a - 8| = 6$
5. $\sqrt{2m + 15} = m$
6. $4q^{-2} + 7q^{-1} = 2$
7. $6n^2 = 7n$
8. $(2x - 5)^2 = 18$

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8–2 Inverse and Joint Variation

Objective: To solve problems involving inverse and joint variation.

Vocabulary

Inverse variation A function defined by an equation of the form $xy = k$ or $y = \frac{k}{x}$ $(x \neq 0, k \neq 0)$. The constant k is called the *constant of variation* or *constant of proportionality*. We say that *y varies inversely as x*, or *y is inversely proportional to x*, because if x increases, y decreases, and if x decreases, y increases.

Joint variation When a quantity varies directly as the product of two or more other quantities, the variation is called joint variation. Example: If m varies jointly as n and p, then $m = knp$ for some nonzero constant k. Another way to state this relationship is "m is jointly proportional to n and p."

Example 1 If y is inversely proportional to x, and $y = 8$ when $x = 6$, find x when $y = 15$.

Solution First find k and write an equation of the inverse variation.

$xy = k$	Start with the general equation.
$(6)(8) = k$	Substitute $x = 6$ and $y = 8$.
$k = 48$	Solve for k.

$\therefore$ an equation of the inverse variation is $xy = 48$.

To find x when $y = 15$, substitute this value in $xy = 48$.

$$x(15) = 48$$
$$x = 3.2$$

Solve.

1. If y varies inversely as x, and $y = 5$ when $x = 4$, find x when $y = 10$.
2. If p is inversely proportional to q, and $p = 10$ when $q = 5$, find q when $p = 2$.
3. If a is inversely proportional to b, and $b = 12$ when $a = 8$, find b when $a = 3$.
4. If x varies inversely as the square of y, and $x = 2$ when $y = 12$, find y when $x = 8$.

Example 2 If z varies jointly as x and the square of y, and $z = 12$ when $x = 6$ and $y = 2$, find z when $x = 10$ and $y = 3$.

Solution First find k and write an equation of the joint variation. Remember it's a *direct* variation.

$$z = kxy^2$$
$$12 = k(6)(2)^2$$
$$12 = 24k$$
$$k = \frac{1}{2}$$

$\therefore$ an equation of the joint variation is $z = \frac{1}{2}xy^2$.

Then, substitute $x = 10$ and $y = 3$ in this equation.

$$z = \frac{1}{2}xy^2$$
$$z = \frac{1}{2}(10)(3)^2$$
$$z = 45$$

8–2 *Inverse and Joint Variation* *(continued)*

Solve.

5. If x varies jointly as y and z, and $x = 100$ when $y = 20$ and $z = 10$, find x when $y = 60$ and $z = 30$.

6. If a is jointly proportional to b and c, and $a = 48$ when $b = 6$ and $c = 4$, find c when $a = 540$ and $b = 18$.

7. If I varies jointly as p and r, and $I = 14$ when $p = 100$ and $r = 0.07$, find p when $I = 48$ and $r = 0.08$.

8. If x varies jointly as y and the square root of z, and $x = 20$ when $y = 5$ and $z = 9$, find x when $y = 6$ and $z = 25$.

Example 3 The intensity of light, measured in *lux*, is inversely proportional to the square of the distance between the light source and the object illuminated. A light meter 6.4 m from a light source registers 30 lux. What intensity would it register 16 m from the light source?

Solution Let I = the intensity of light in lux, and d = the distance in meters of the illuminated object from the light source. I varies inversely as the square of d. That is,

$$Id^2 = k.$$

Find k when $I = 30$ lux and $d = 6.4$ m.

$$30(6.4)^2 = k$$
$$k = 1228.8$$

$\therefore$ an equation of the variation is $Id^2 = 1228.8$.

Then, to find I when $d = 16$ m, substitute this value in $Id^2 = 1228.8$.

$$I(16)^2 = 1228.8$$
$$I = 4.8$$

$\therefore$ the intensity is 4.8 lux.

Solve.

9. A light meter 5.4 m from a light source registers 20 lux. What intensity would it measure 1.8 m from the light source?

10. The frequency of a radio signal varies inversely as the wavelength. A signal of frequency 1250 kilohertz (kHz) has a wavelength of 240 m. What frequency has a signal of wavelength 300 m?

11. The volume of a cylinder is jointly proportional to the height and the square of the radius of a base. A cylinder of height 12 cm and base radius 3 cm has volume 108π cm^3. Find the radius of the base of a cylinder if the height is 2 cm and the volume is 72π cm^3.

12. The stretch in a wire under a given tension varies directly as the length of the wire and inversely as the square of its diameter. If the length and the diameter of a wire are both doubled, what is the effect on the stretch of the wire?

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8–3 Dividing Polynomials

Objective: To divide one polynomial by another polynomial.

Vocabulary

Dividend A quantity to be divided. In long division, the number *under* the division symbol is the dividend. In a fraction, the numerator is the dividend.

Divisor The quantity by which another quantity is divided. In long division, the number *outside* the division symbol is the divisor. In a fraction, the denominator is the divisor.

Division algorithm (rule) $\dfrac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \dfrac{\text{Remainder}}{\text{Divisor}}$

or $\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$

CAUTION Before using long division, always arrange the terms of both dividend and divisor in order of decreasing degree. Also, insert any "missing" terms by using zero as a coefficient. Example: $x - 3x^3 - 5$ becomes $-3x^3 + 0x^2 + x - 5$.

Example 1 Divide: $\dfrac{3x + 2x^3 - 5}{x + 2}$

Solution

1. Rewrite the problem in the form you would use to do long division.
2. Divide the first term of the dividend by the first term of the divisor. Write this monomial above the line as the first term of the quotient.
3. Multiply by the new quotient term.
4. Subtract.
5. Repeat Steps 2–4 until the remainder is 0 or the degree of the remainder is less than the degree of the divisor.

$$
\begin{array}{rrl}
 & 2x^2 - 4x + 11 & \longleftarrow \textbf{quotient} \\
\textbf{divisor} \longrightarrow x + 2 & \overline{)\,2x^3 + 0x^2 + 3x - 5} & \longleftarrow \textbf{dividend} \\
 & \underline{2x^3 + 4x^2} \qquad\qquad & \longleftarrow \text{Multiply } x + 2 \text{ by } 2x^2. \\
 & -4x^2 + 3x \qquad & \longleftarrow \text{Subtract.} \\
 & \underline{-4x^2 - 8x} \qquad & \longleftarrow \text{Multiply } x + 2 \text{ by } -4x. \\
 & 11x - 5 & \longleftarrow \text{Subtract.} \\
 & \underline{11x + 22} & \longleftarrow \text{Multiply } x + 2 \text{ by } 11. \\
\textbf{remainder} \longrightarrow & -27 & \longleftarrow \text{Subtract.}
\end{array}
$$

$$\therefore \underbrace{\frac{2x^3 + 3x - 5}{x + 2}}_{\frac{\text{Dividend}}{\text{Divisor}}} = \underbrace{2x^2 - 4x + 11}_{\text{Quotient}} + \underbrace{\frac{-27}{x + 2}}_{\frac{\text{Remainder}}{\text{Divisor}}}$$

Check: To check the result, use the following form of the division algorithm:

$$
\begin{array}{rcl}
\text{Quotient} \times \text{Divisor} \quad + \quad \text{Remainder} & = & \text{Dividend} \\
(2x^2 - 4x + 11)(x + 2) + (-27) & \stackrel{?}{=} & 2x^3 + 3x - 5 \\
2x^3 - 4x^2 + 11x + 4x^2 - 8x + 22 - 27 & \stackrel{?}{=} & 2x^3 + 3x - 5 \\
2x^3 + 3x - 5 & = & 2x^3 + 3x - 5 \quad \checkmark
\end{array}
$$

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8–3 *Dividing Polynomials* *(continued)*

Divide. Additional answers are given at the back of this Answer Key.

1. $\dfrac{x^2 + 8x + 12}{x + 4}$ **2.** $\dfrac{-68 + 17x - x^2}{6 - x}$ **3.** $\dfrac{x^2 - 16x}{x - 4}$

4. $\dfrac{9x^2 - 6x + 1}{3x + 1}$ **5.** $\dfrac{6x^2 - 13x + 6}{2x - 3}$ **6.** $\dfrac{10x^2 + x - 3}{5x + 3}$

7. $\dfrac{2x^3 - 15x + 8}{x - 2}$ **8.** $\dfrac{x^3 - 4x - 8}{x - 2}$ **9.** $\dfrac{6x^3 - 19x^2 + 15}{3x - 5}$

Example 2 Divide: $\dfrac{x^4 + x^2 + 4}{x^2 + x + 1}$

Solution Insert the "missing" terms, $0x^3$ and $0x$, in the dividend.

$$\begin{array}{r|l} & x^2 - x + 1 \\ \hline x^2 + x + 1 & x^4 + 0x^3 + x^2 + 0x + 4 \\ & \underline{x^4 + 1x^3 + x^2} \\ & \quad -x^3 + 0x^2 + 0x \\ & \quad \underline{-x^3 - x^2 - x} \\ & \qquad x^2 + x + 4 \\ & \qquad \underline{x^2 + x + 1} \\ & \qquad\qquad 3 \end{array}$$

$\therefore$ the quotient is $x^2 - x + 1$, and the remainder is 3.

$$\therefore \frac{x^4 + x^2 + 4}{x^2 + x + 1} = x^2 - x + 1 + \frac{3}{x^2 + x + 1}$$

The check is left for you.

Divide.

10. $\dfrac{x^3 - 6x^2 + 12x - 8}{x^2 - 4x + 4}$ **11.** $\dfrac{x^4 - x^3 + 7x + 5}{x^2 + 2x + 1}$

12. $\dfrac{3x^4 + x^3 - 2x + 7}{x^2 - x + 1}$ **13.** $\dfrac{4x^4 + 5x^2 + 12x + 1}{2x^2 - x - 3}$

14. $\dfrac{6x^4 + 5x^3 - 9x^2 + 5}{3x^2 + x - 2}$ **15.** $\dfrac{10x^4 - 2x^3 + x^2 + x - 3}{2x^2 - 1}$

Mixed Review Exercises

In Exercises 1–4, assume that y varies directly as x.

1. If $y = 10$ when $x = 4$, find y when $x = 5$.

2. If $y = 2$ when $x = 0.5$, find y when $x = 6$.

3. If $y = 7$ when $x = \sqrt{3}$, find y when $x = \sqrt{15}$.

4. If $y = 4.2$ when $x = 0.4$, find y when $x = 2.5$.

5–8. Solve 1–4, assuming that y varies inversely as x.

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8–4 Synthetic Division

Objective: To use synthetic division to divide a polynomial by a first-degree binomial.

Vocabulary

Synthetic division An efficient way to divide a polynomial in x by a binomial of the form $x - c$. It uses only the coefficients of the polynomials involved.

Example 1 Use synthetic division to divide $x^4 - 3x^3 + 2x^2 - 4$ by $x + 2$.

Solution

1. Arrange the terms of the polynomial in order from the largest to the smallest exponent: $x^4 - 3x^3 + 2x^2 + 0x - 4$. Then write the coefficients of the terms in a row. The coefficient of x^4 is 1. Use a 0 as the coefficient of any power of x that is "missing."
2. Write the constant term c of the binomial divisor $x - c$ to the left of the coefficients. *Note:* Since $x + 2 = x - (-2)$, you use -2 for c.
3. Bring down the first coefficient.
4. Multiply what you've brought down by c and write the result below the next coefficient.
5. Add.
6. Repeat Steps 4 and 5 for every coefficient.

Step 2

$$\begin{array}{l|rrrrr|l}
\underline{-2} & 1 & -3 & 2 & 0 & -4 & \longleftarrow \textit{Step 1} \\
 & \downarrow & -2 & 10 & -24 & 48 & \\
\hline
\textit{Step 3} \longrightarrow & 1 & -5 & 12 & -24 & 44 &
\end{array}$$

Steps 4, 5   *Remainder*

The resulting numbers 1, -5, 12, and -24 are the coefficients of the quotient. The degree of the quotient will be 1 less than the degree of the dividend, or 3. The quotient is $1x^3 - 5x^2 + 12x - 24$, and the remainder is 44.

$$\therefore \frac{x^4 - 3x^3 + 2x^2 - 4}{x + 2} = x^3 - 5x^2 + 12x - 24 + \frac{44}{x + 2}$$

Example 2 Express as a polynomial (a) the divisor, (b) the dividend, (c) the quotient, and (d) the remainder for the synthetic division shown below.

$$\begin{array}{r|rrrr|r}
\underline{3} & 1 & 2 & -10 & 0 & -41 \\
 & & 3 & 15 & 15 & 45 \\
\hline
 & 1 & 5 & 5 & 15 & 4
\end{array}$$

(*Solution is on the next page.*)

8–4 Synthetic Division *(continued)*

Solution You need to put the variables back in place.

a. The divisor has the form $x - c$, and $c = 3$. Thus, the divisor is $x - 3$.

b. There are 5 coefficients in the top row. The coefficient -41 is a constant. The dividend is the *fourth*-degree polynomial $x^4 + 2x^3 - 10x^2 + 0x - 41$, or $x^4 + 2x^3 - 10x^2 - 41$.

c. The first four coefficients in the bottom row form the quotient. The coefficient 15 is a constant. Thus, the quotient is the *third*-degree polynomial $x^3 + 5x^2 + 5x + 15$.

d. The remainder is the last number in the bottom row, or 4.

For each synthetic division shown below, express as a polynomial (a) the divisor, (b) the dividend, (c) the quotient, and (d) the remainder. Use x as the variable.

1.
$$\begin{array}{r|rrrr} -3 & 1 & 5 & 7 & 3 \\ & & -3 & -6 & -3 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

2.
$$\begin{array}{r|rrrrr} 4 & 1 & -6 & 0 & 18 & -10 \\ & & 4 & -8 & -32 & -56 \\ \hline & 1 & -2 & -8 & -14 & -66 \end{array}$$

Example 3 Divide using synthetic division: $\dfrac{x^4 - 16}{x - 2}$

Solution Use 2 for c. Insert zeros for the missing terms x^3, x^2, and x in the dividend.

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & 0 & 0 & -16 \\ & & 2 & 4 & 8 & 16 \\ \hline & 1 & 2 & 4 & 8 & 0 \end{array}$$

The quotient is $1x^3 + 2x^2 + 4x + 8$, and the remainder is 0.

$$\therefore \frac{x^4 - 16}{x - 2} = x^3 + 2x^2 + 4x + 8$$

Divide using synthetic division.

3. $\dfrac{x^3 + 2x^2 - 2x - 4}{x + 1}$

4. $\dfrac{x^3 + 2x^2 - 2x - 1}{x - 1}$

5. $\dfrac{x^3 + x^2 - 8x - 12}{x + 3}$

6. $\dfrac{2x^3 - 5x^2 + 4x - 3}{x - 2}$

7. $\dfrac{x^4 + 2x^3 + 3x^2 - x - 3}{x + 1}$

8. $\dfrac{3x^4 + 10x^3 - 7x^2 + 5x + 6}{x + 4}$

9. $\dfrac{x^4 + 3x^2 - x - 5}{x + 2}$

10. $\dfrac{2x^4 - 7x^3 - x + 10}{x - 3}$

11. $\dfrac{x^5 - 32}{x - 2}$

12. $\dfrac{x^7 - 1}{x - 1}$

13. $\dfrac{x^6 + 2x^3 - 7x^2 + 8x + 4}{x + 2}$

14. $\dfrac{x^6 - 3x^5 + 14x + 4}{x - 2}$

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8-5 The Remainder and Factor Theorems

Objective: To use the remainder and factor theorems to find factors of polynomials and to solve polynomial equations.

Vocabulary

Synthetic substitution Another name for synthetic division.

Remainder theorem Let $P(x)$ be a polynomial of positive degree n, with real or complex coefficients. For any number c, $P(x) = Q(x) \cdot (x - c) + P(c)$, where $Q(x)$ is a polynomial of degree $n - 1$. In other words, the *remainder* when the polynomial $P(x)$ is divided by the binomial $x - c$ is equal to $P(c)$.

Factor theorem The polynomial $P(x)$ has $x - r$ as a factor if and only if r is a root of the equation $P(x) = 0$.

Depressed equation The equation $Q(x) = 0$, where $Q(x)$ is the quotient when $P(x)$ is divided by one of its factors. The roots of the depressed equation are also roots of the equation $P(x) = 0$.

Example 1 Use synthetic substitution to find the value $P(-3)$ for the polynomial $P(x) = x^4 + 4x^3 - x^2 - 10x + 12$.

Solution By the remainder theorem, $P(-3)$ is equal to the remainder when $P(x)$ is divided by $x - (-3)$. Using synthetic substitution, divide by -3.

$$\begin{array}{r|rrrrr} c \longrightarrow -3 & 1 & 4 & -1 & -10 & 12 \\ & & -3 & -3 & 12 & -6 \\ \hline & 1 & 1 & -4 & 2 & 6 \longleftarrow P(c) \end{array} \qquad \therefore P(-3) = 6$$

Check: You can check the answer by directly calculating $P(-3)$.

$$\begin{aligned} P(-3) &= (-3)^4 + 4(-3)^3 - (-3)^2 - 10(-3) + 12 \\ &= 81 - 108 - 9 + 30 + 12 = 6 \quad \surd \end{aligned}$$

Use synthetic substitution to find $P(c)$ for the given polynomial $P(x)$ and the given number c.

1. $P(x) = x^3 - 3x^2 - 4x + 6;\ c = 3$
2. $P(x) = x^3 + 2x^2 - 6x - 4;\ c = -2$
3. $P(x) = 12 - 17x - 10x^2 - x^3;\ c = -3$
4. $P(x) = x^4 - 4x^2 + x + 4;\ c = -2$
5. $P(x) = 10 - 16x^2 + 6x^3;\ c = -\frac{1}{3}$
6. $P(x) = 4x^3 + 2x^2 - 4x + 6;\ c = -\frac{3}{2}$

Example 2 Determine whether the binomial is a factor of the given polynomial.

a. $x - 3;\ P(x) = 2x^3 - 4x^2 - 7x + 3$ **b.** $z - i;\ P(z) = z^3 + z^2 + z + 1$

Solution

a. By the factor theorem, if $P(3) = 0$, then $x - 3$ is a factor. Find $P(3)$.

$$\begin{aligned} P(3) &= 2(3)^3 - 4(3)^2 - 7(3) + 3 \\ &= 54 - 36 - 21 + 3 = 0 \end{aligned}$$

$\therefore x - 3$ is a factor of $P(x)$.

b. By the factor theorem, if $P(i) = 0$, then $z - i$ is a factor. Find $P(i)$.

$$\begin{aligned} P(i) &= i^3 + i^2 + i + 1 \\ &= -i + -1 + i + 1 = 0 \end{aligned}$$

$\therefore z - i$ is a factor of $P(z)$.

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8–5 The Remainder and Factor Theorems *(continued)*

Use the factor theorem to determine whether the binomial is a factor of the given polynomial.

7. $x - 1$; $P(x) = x^6 - x^4 + x^2 - 1$

8. $y + 1$; $P(y) = y^7 + y^4 + y$

9. $n - 2$; $P(n) = n^4 - 2n^3 - 3n + 6$

10. $z + \sqrt{5}$; $P(z) = z^4 - 2z^3 - 3z^2 + 10z - 10$

11. $y + 2$; $P(y) = y^4 - y^2 + 4y + 2$

12. $x - i$; $P(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

Example 3 Solve $x^3 + x^2 - 11x - 3 = 0$ given that 3 is a root.

Solution Find the depressed equation by dividing $P(x) = x^3 + x^2 - 11x - 3$ by $x - 3$.

$$\begin{array}{r|rrrr} 3 & 1 & 1 & -11 & -3 \\ & & 3 & 12 & 3 \\ \hline & 1 & 4 & 1 & 0 \end{array}$$

So $x^3 + x^2 - 11x - 3 = (x - 3)(x^2 + 4x + 1)$.

The *depressed equation* is $x^2 + 4x + 1 = 0$. Solve the depressed equation.

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(1)}}{2(1)} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

$\therefore$ the solution set is $\{3, -2 + \sqrt{3}, -2 - \sqrt{3}\}$.

In Exercises 13–15, a root of the equation is given. Solve the equation.

13. $x^3 + 3x^2 - 18x - 40 = 0$; 4

14. $x^3 - 5x - 2 = 0$; -2

15. $x^3 - x^2 + 3x + 5 = 0$; -1

Example 4 Find a polynomial equation that has 3, $3i$, and $-3i$ as roots.

Solution By the factor theorem, the required polynomial must have factors $(x - 3)$, $(x - 3i)$, and $(x - (-3i))$. Therefore, a solution is

$$(x - 3)(x - 3i)(x + 3i) = 0, \text{ or}$$
$$x^3 - 3x^2 + 9x - 27 = 0.$$

Find a polynomial equation that has the given numbers as roots.

16. $1, -2, 3$

17. $-1, 1, -2$

18. $0, -2, 4$

19. $-4, 2i, -2i$

Mixed Review Exercises

Divide.

1. $\dfrac{x^2 + 5x - 2}{x + 1}$

2. $\dfrac{y^3 + 1}{y^2 - y + 1}$

3. $\dfrac{6a^2 - 9a + 5}{2a - 1}$

4. $\dfrac{4c^3 + 7c + 4}{2c^2 - c + 4}$

Without solving, determine whether the roots of each equation are rational, irrational, or imaginary.

5. $2x^2 + 7x + 8 = 0$

6. $2x^2 - 7x + 6 = 0$

7. $2x^2 - 7x - 6 = 0$

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8–6 Some Useful Theorems

Objective: To find or solve a polynomial equation with real coefficients and positive degree *n* by using these facts:

1. There are exactly *n* roots.
2. The imaginary roots occur in conjugate pairs.
3. Descartes' rule of signs gives information about the number of positive and negative real roots.

Vocabulary

Degree of a polynomial equation The degree of a polynomial equation of the form $P(x) = 0$ is the degree of $P(x)$. Example: The degree of $x^3 - 2x + 7 = 0$ is 3.

Theorem Every polynomial equation with complex coefficients and positive degree n has exactly n roots (provided that a multiple root is counted as many times as it appears). Examples: $4x^3 + x + 5$ has exactly 3 roots; $x^5 - 5ix^4 + 9x^2 - 2ix + 12$ has exactly 5 roots.

Conjugate root theorem If a polynomial equation with *real* coefficients has $a + bi$ as a root (a and b real, $b \neq 0$), then $a - bi$ is also a root.

Variation in sign A change of sign from one term of a polynomial to the next. Missing terms are ignored. Example: $3 + x^5 + 2x - x^2 = x^5 - x^2 + 2x + 3$ has two variations in sign.

Descartes' rule of signs Let $P(x)$ be a simplified polynomial with real coefficients and terms arranged in decreasing degree of x.

1. The number of *positive* real roots of $P(x) = 0$ equals the number of variations of sign of $P(x)$ or is fewer than this number by an even integer. Example: $P(x) = x^5 - x^2 + 2x + 3$ has two variations in sign, so the equation $x^5 - x^2 + 2x + 3 = 0$ has either two or zero positive real roots.
2. The number of *negative* real roots of $P(x) = 0$ equals the number of variations in sign of $P(-x)$ or is fewer than this number by an even integer. Example: $P(-x) = (-x)^5 - (-x)^2 + 2(-x) + 3 = -x^5 - x^2 - 2x + 3$ has one variation in sign. Thus, the equation $-x^5 - x^2 + 2x + 3 = 0$ has only one negative real root.

Example 1 Find a cubic equation with integral coefficients that has 3 and $2 - i$ as roots.

Solution The third root must be the conjugate of $2 - i$, which is $2 + i$. An equation with these three roots is:

$$
\begin{aligned}
(x - 3)[x - (2 - i)][x - (2 + i)] &= 0\\
(x - 3)[x^2 - (2 - i)x - (2 + i)x + (2 - i)(2 + i)] &= 0\\
(x - 3)(x^2 - 4x + 5) &= 0\\
x^3 - 3x^2 - 4x^2 + 12x + 5x - 15 &= 0\\
x^3 - 7x^2 + 17x - 15 &= 0
\end{aligned}
$$

Find a cubic equation with integral coefficients that has the given numbers as roots.

1. $2, i$ **2.** $5, i\sqrt{3}$ **3.** $-1, 1 - i$ **4.** $-3, 2 + 3i$

8–6 Some Useful Theorems *(continued)*

Example 2 Solve $x^3 - 5x^2 + 11x - 15 = 0$ given that $1 - 2i$ is a root.

Solution Let $P(x) = x^3 - 5x^2 + 11x - 15$.

1. If $1 - 2i$ is a root of $P(x) = 0$, then $1 + 2i$ is also a root, by the conjugate root theorem. Therefore, $x - (1 - 2i)$ and $x - (1 + 2i)$ are factors of $P(x)$. Their product, $[x - (1 - 2i)][x - (1 + 2i)] = x^2 - 2x + 5$, is also a factor of $P(x)$; that is $P(x) = (x^2 - 2x + 5) \cdot Q(x)$.
2. Long division will show that $Q(x) = P(x) \div (x^2 - 2x + 5) = x - 3$.
3. The depressed equation, $x - 3 = 0$, has root 3.

$\therefore$ the solution set of the given equation is $\{1 - 2i, 1 + 2i, 3\}$.

In each exercise all but one of the equation's roots are given. Find the remaining root. Check your answer by substituting it for x in the equation.

5. $x^3 + 2x^2 + 2x + 40 = 0$; -4 and $1 + 3i$

6. $x^4 - 8x^3 + 22x^2 - 56x + 105 = 0$; 3, 5, $i\sqrt{7}$

In each exercise a root of the equation is given. Solve the equation.

7. $x^3 - 10x^2 + 35x - 38 = 0$; $4 + i\sqrt{3}$

8. $x^3 - 5x^2 + 2x - 10 = 0$; $-i\sqrt{2}$

9. $x^4 - 2x^3 - 4x^2 + 14x - 5 = 0$; $2 + i$

10. $x^4 + 8x^3 + 19x^2 + 2x - 30 = 0$; $-3 - i$

Example 3 List the possibilities for the nature of the roots (positive real, negative real, and imaginary) for the equation $P(x) = 0$, where

$$P(x) = -3x^4 + x^3 - 6x^2 + 3x + 2.$$

Solution

1. $P(x)$ has three variations in sign, so by the first part of Descartes' rule, the number of positive real roots of $P(x) = 0$ is 3 or 1.
2. $P(-x) = -3(-x)^4 + (-x)^3 - 6(-x)^2 + 3(-x) + 2$
   $= -3x^4 - x^3 - 6x^2 - 3x + 2$
   $P(-x)$ has one variation in sign, so by the second part of Descartes' rule the number of negative roots of $P(x) = 0$ is 1.
3. Since $P(x)$ has four roots in all, by the first theorem on the previous page, there are only two possibilities. These are listed in the chart below.

Number of positive real roots	Number of negative real roots	Number of imaginary roots
3	1	0
1	1	2

List the possibilities for the nature of the roots of each equation.

11. $x^3 - 2x^2 + 3x + 3 = 0$

12. $x^4 + x^3 - 16 = 0$

13. $x^4 - 5x^3 - 2x^2 + x + 1 = 0$

14. $x^6 - x^4 + x - 1 = 0$

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8–7 Finding Rational Roots

Objective: To find rational roots of polynomial equations with integral coefficients.

Vocabulary

Rational root theorem Suppose that a polynomial equation with integral coefficients has the root $\frac{h}{k}$, where h and k are relatively prime integers (that is, integers whose GCF is 1). Then h must be a factor of the constant term of the polynomial and k must be a factor of the coefficient of the highest-degree term.

Example 1 Solve the equation $2x^4 + 5x^3 + 3x^2 + 15x - 9 = 0$ by first using the rational root theorem to find any rational roots.

Solution Since h is ± 1, ± 3, or ± 9 (the factors of 9), and k is ± 1 or ± 2 (the factors of 2), the only possible rational roots are

$$\pm 1,\ \pm 3,\ \pm 9,\ \pm\frac{1}{2},\ \pm\frac{3}{2},\ \text{and}\ \pm\frac{9}{2}.$$

You should try the integers first. Check 1 and -1 by substitution. You will find that neither is a root. Use synthetic substitution to try the other possibilities.

$$\begin{array}{r|rrrrr} 3 & 2 & 5 & 3 & 15 & -9 \\ & & 6 & 33 & 108 & 369 \\ \hline & 2 & 11 & 36 & 123 & 360 \end{array}$$

3 is *not* a root.

$$\begin{array}{r|rrrrr} -3 & 2 & 5 & 3 & 15 & -9 \\ & & -6 & 3 & -18 & 9 \\ \hline & 2 & -1 & 6 & -3 & 0 \end{array}$$

Since the remainder is 0, -3 *is* a root.

Once you have a root, use the depressed equation ($2x^3 - x^2 + 6x - 3 = 0$) to find the remaining roots. The depressed equation has ± 1, ± 3, $\pm\frac{1}{2}$, and $\pm\frac{3}{2}$ as possible rational roots. You eliminated 1, -1, and 3 using the original equation. Since -3 may be a double root, it is still a possibility, along with $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$.

$$\begin{array}{r|rrrr} -3 & 2 & -1 & 6 & -3 \\ & & -6 & 21 & -81 \\ \hline & 2 & -7 & 27 & -84 \end{array}$$

-3 is *not* a double root.

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & 6 & -3 \\ & & 1 & 0 & 3 \\ \hline & 2 & 0 & 6 & 0 \end{array}$$

$\frac{1}{2}$ *is* a root.

The second depressed equation is $2x^2 + 6 = 0$ (or $x^2 + 3 = 0$). If you rewrite this quadratic equation as $x^2 = -3$, and take the square root of both sides, you obtain the solutions $x = \pm i\sqrt{3}$.

$\therefore$ the original equation has solution set $\left\{-3, \frac{1}{2}, i\sqrt{3}, -i\sqrt{3}\right\}$.

Find any rational roots. If the equation has at least one rational root, solve it completely.

1. $x^3 + 2x^2 - 13x + 10 = 0$

2. $x^3 - 4x^2 - x + 4 = 0$

3. $x^3 - 3x^2 - 5x + 15 = 0$

4. $x^4 + x^2 + 9 = 0$

8–7 Finding Rational Roots *(continued)*

Find any rational roots. If the equation has at least one rational root, solve it completely.

5. $x^4 - 4x^3 + 6x^2 - 12x + 9 = 0$
6. $x^4 - x^3 - 6x^2 + 14x - 12 = 0$
7. $4x^4 - 8x^3 - 9x^2 + 2x + 2 = 0$
8. $3x^4 - 5x^3 + 10x^2 - 20x - 8 = 0$

Example 2 Show that $\sqrt{8}$ is irrational.

Solution $\sqrt{8}$ is a root of the equation $x^2 - 8 = 0$. The only possible rational roots of this equation are ± 1, ± 2, ± 4, and ± 8. If you substitute these numbers into the equation, you will find that none of them satisfy it. Since $x^2 - 8 = 0$ has no rational roots, $\sqrt{8}$ is irrational.

Use the method of Example 2 to show that the following numbers are irrational.

9. $\sqrt{2}$
10. $\sqrt{10}$
11. $\sqrt[3]{-9}$
12. $\sqrt[3]{5}$
13. $\sqrt[4]{9}$
14. $\sqrt[5]{-12}$

Example 3 Verify that $\sqrt{3} - \sqrt{2}$ is a root of $x^4 - 10x^2 + 1 = 0$ and use this fact to show that the number is irrational.

Solution To verify that $\sqrt{3} - \sqrt{2}$ is a root, substitute it into the equation.

$$(\sqrt{3} - \sqrt{2})^4 - 10(\sqrt{3} - \sqrt{2})^2 + 1 \stackrel{?}{=} 0$$
$$49 - 20\sqrt{6} - 10(5 - 2\sqrt{6}) + 1 \stackrel{?}{=} 0$$
$$49 - 20\sqrt{6} - 50 + 20\sqrt{6} + 1 \stackrel{?}{=} 0$$
$$0 = 0 \quad \checkmark$$

$\therefore$ $\sqrt{3} - \sqrt{2}$ is a root, or solution, of the equation $x^4 - 10x^2 + 1 = 0$.

By the rational root theorem, the only possible rational roots of the equation are ± 1. If you substitute these numbers into the equation, you will find that neither of them is a solution. Since the equation has no rational roots, $\sqrt{3} - \sqrt{2}$ is an irrational number.

Verify that the given number is a root of the equation and use this fact to show that the number is irrational.

15. $\sqrt{2} + \sqrt{5}$; $x^4 - 14x^2 + 9 = 0$
16. $\sqrt{3} - \sqrt{6}$; $x^4 - 18x^2 + 9 = 0$

Mixed Review Exercises

Solve each equation completely. In Exercises 7 and 8, one root is given.

1. $5(2 - x) + 3 = 7 - 4x$
2. $2x^2 - 2x + 3 = 0$
3. $\sqrt{2x + 7} = x + 2$
4. $\dfrac{1}{x} + \dfrac{x}{x - 1} + \dfrac{1}{x - x^2} = 0$
5. $8x^{-2} + 4x^{-1} - 12 = 0$
6. $x^3 - 6x^2 + 13x - 10 = 0$
7. $2x^3 - 11x^2 + 12x + 9 = 0$; 3
8. $x^4 - x^2 - 6 = 0$; $i\sqrt{2}$

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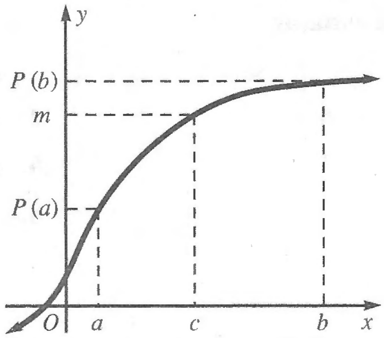
8–8 Approximating Irrational Roots

Objective: To approximate the real roots of a polynomial equation $P(x) = 0$ by using the graph of $y = P(x)$.

Vocabulary

Intermediate-value theorem
If P is a polynomial function with real coefficients, and m is any number between $P(a)$ and $P(b)$, then there is at least one number c between a and b for which $P(c) = m$.



CAUTION When using a graph to approximate the roots of a function, keep in mind that you can only approximate *real* roots by this method, not imaginary roots.

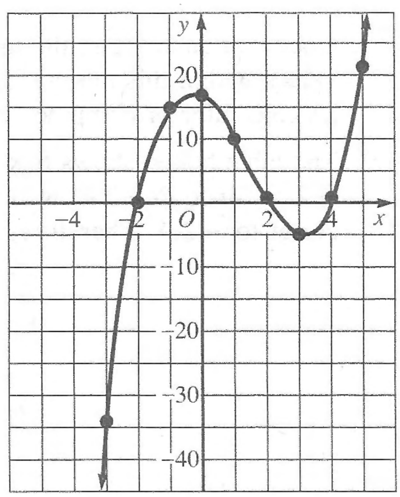
Example 1 **a.** Graph the equation $y = x^3 - 4x^2 - 4x + 17$.

b. Use the graph to estimate to the nearest unit the roots of the equation $x^3 - 4x^2 - 4x + 17 = 0$.

Solution **a.** You may wish to use a computer or graphing calculator if one is available. Otherwise make a table of values, plot the corresponding points, and join the points with a smooth unbroken curve. When making your table, apply Descartes' rule of signs to choose an appropriate range of values for x. When making your graph, use a smaller scale on the y-axis than on the x-axis to accommodate the greater range of y-values.

$y = x^3 - 4x^2 - 4x + 17$

x	y
-3	-34
-2	1
-1	16
0	17
1	10
2	1
3	-4
4	1
5	22



b. The roots of the equation are the x-coordinates of the points where the graph intersects the x-axis. To the nearest unit, the roots are -2, 2, and 4.
Note: Roots occur in intervals where the sign of the y-value changes.

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8–8 *Approximating Irrational Roots* *(continued)*

For each polynomial P, draw a graph to approximate to the nearest unit the real root(s) of the equation $P(x) = 0$. You may wish to use a computer or a graphing calculator.

1. $P(x) = x^3 - 6$

2. $P(x) = x^3 + 9$

3. $P(x) = x^3 - x^2 + 5x + 6$

4. $P(x) = -x^3 - x + 8$

5. $P(x) = x^3 - 2x^2 + x - 3$

6. $P(x) = 2x^3 - 3x^2 - 3x + 3$

7. $P(x) = x^4 + x^3 - 14$

8. $P(x) = x^4 - 2x^3 - 4x^2 + 4x$

Example 2 Approximate to the nearest tenth the real zero of the function $P(x) = x^3 + 4x^2 + 10x + 15$.

Solution Recall that the zeros of P are the roots of the polynomial equation $P(x) = 0$. By Descartes' rule, $P(x) = 0$ has no positive roots, so make a table using nonpositive values of x. Begin with integers, and look for the interval where the value of $P(x)$ changes sign.

The table below shows that $P(x)$ changes sign when x is in the interval between -3 and -2. By the intermediate-value theorem, there is a value r between -3 and -2 such that $P(r) = 0$.

	x	$P(x)$	
	0	15	
	-1	8	
	-2	3	
$r \rightarrow$	-3	-6	$\leftarrow 0$

Once you have found the interval that contains the root, make a table using values within this interval to find the answer to the nearest tenth. (A calculator or computer will be very helpful.)

The table below shows that $P(x)$ changes sign when x is between -2.4 and -2.5. Since $P(-2.4)$ is closer to 0 than $P(-2.5)$, you may assume that r is closer to -2.4. Therefore, to the nearest tenth, the real zero of P is -2.4.

	x	$P(x)$	
	-2.1	2.379	
	-2.2	1.712	
	-2.3	0.993	
	-2.4	0.216	
$r \rightarrow$	-2.5	-0.625	$\leftarrow 0$

9–13. In Exercises 1–5 above, each polynomial equation $P(x) = 0$ has only one real root. Approximate it to the nearest tenth.

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8–9 Linear Interpolation

Objective: To use linear interpolation to find values not listed in a given table of data.

Vocabulary

Linear interpolation A process used to approximate values not given in a table of data. The method is based upon the assumption that within a given interval, the change in one value is proportional to the change in the other value.

Example 1 Use linear interpolation and the table to find these values to the nearest integer:

a. the approximate number of bachelor's degrees awarded in computer science in 1971

b. the approximate year that 10,000 bachelor's degrees were awarded in computer science

Year	Bachelor's degrees awarded in computer science (U.S.)
1968	459
1972	3,402
1976	5,700
1980	11,154
1984	32,172

Solution

a. 1971 is three quarters of the way from 1968 to 1972. Assume that the value you're looking for is similarly three quarters of the way from 459 to 3402. Write a proportion to find three quarters of the difference between 459 and 3402, and then add this amount to 459 to find d, the approximate number of degrees awarded.

		Year	Bachelor's degrees awarded in computer science (U.S.)		
4	3	1968	459	x	2943
		1971	d		
		1972	3402		

$$\frac{x}{2943} = \frac{3}{4}$$

$$x = \frac{3}{4}(2943) \approx 2207$$

$$d = 459 + x$$

$$\approx 459 + 2207 = 2666$$

$\therefore$ the number of bachelor's degrees awarded in computer science in 1971 was about 2666.

b. Finding a value in the first column to correspond with a given value in the second column is called *inverse interpolation*. While the name of the process is different, the method is basically the same.

		Year	Bachelor's degrees awarded in computer science (U.S.)		
4	x	1976	5,700	4300	5454
		y	10,000		
		1980	11,154		

$$\frac{x}{4} = \frac{4300}{5454}$$

$$x = 4\left(\frac{4300}{5454}\right) \approx 3$$

$$y = 1976 + x$$

$$\approx 1976 + 3 = 1979$$

$\therefore$ there were 10,000 bachelor's degrees awarded in computer science in 1979.

8–9 Linear Interpolation *(continued)*

Use linear interpolation and the table in Example 1 to find the approximate number of bachelor's degrees awarded in computer science in each of the following years.

1. 1975 **2.** 1981 **3.** 1978 **4.** 1969

Use inverse interpolation to approximate the year in which the given number of bachelor's degrees were awarded in computer science.

5. 1930 **6.** 7080 **7.** 27,000 **8.** 3970

Example 2 The table gives the temperature in degrees Fahrenheit on a spring day in Boston, Massachusetts.

a. Approximate the temperature at 3:40 P.M.

b. At about what time was the temperature 40°?

Time (P.M.)	Temperature
1:00	68°
2:00	66°
3:00	63°
4:00	59°
5:00	53°
6:00	45°
7:00	39°

Solution

a. The temperature values in the table *decrease* as time *increases*. As a result, you will have to *subtract* the change in temperature at the last step.

		Time (P.M.)	Temperature		
60	40	3:00	63°	x	4
		3:40	t		
		4:00	59°		

$$\frac{x}{4} = \frac{40}{60}$$

$$x = 4\left(\frac{40}{60}\right) \approx 2.7°$$

$$t = 63 - x$$

$$\approx 63 - 2.7 = 60.3°$$

∴ the temperature at 3:40 was about 60.3°.

b. In this part of the problem, since time is increasing, you will have to *add* the change in time at the last step. When you do this, be sure not to confuse minutes and hours.

		Time (P.M.)	Temperature		
60	x	6:00	45°	5	6
		t	40°		
		7:00	39°		

$$\frac{x}{60} = \frac{5}{6}$$

$$x = 60\left(\frac{5}{6}\right) = 50 \text{ min}$$

∴ the temperature was 40° at about 6:50 P.M.

Use linear interpolation and the table in Example 2 to approximate the temperature at each given time.

9. 4:20 P.M. **10.** 5:45 P.M. **11.** 1:15 P.M. **12.** 6:24 P.M.

Use inverse interpolation and the table in Example 2 to find the approximate time associated with the given temperature.

13. 55° **14.** 44° **15.** 62° **16.** 49°