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13 Trigonometric Graphs; Identities

13–1 Radian Measure

Objective: To use radians to measure angles.

Vocabulary

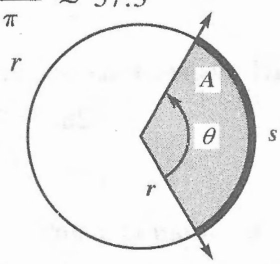
Radian A unit for measuring angles. If a circle of radius r is centered at the vertex of angle θ, then the radian measure of θ is defined as the ratio of s, the length of the arc intercepted by θ, to r: $\theta = \frac{s}{r}$.

Conversion between radian and degree measure Since $180° = \pi$ radians,

$$1° = \frac{\pi}{180} \text{ radians} \approx 0.0175 \text{ radians} \quad \text{and} \quad 1 \text{ radian} = \frac{180°}{\pi} \approx 57.3°$$

Formulas for arc length and area of a sector If a circle of radius r is centered at the vertex of an angle θ measured in radians, then the length of the arc intercepted by θ is given by $s = r\theta$ and the area of the related sector is given by

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}rs.$$



Example 1	Express in radians: **a.** 75°	**b.** −225°
Solution	**a.** $75° = 75 \cdot \frac{\pi}{180}$ radians $= \frac{5\pi}{12}$ radians	**b.** $-225° = -225 \cdot \frac{\pi}{180}$ radians $= -\frac{5\pi}{4}$ radians

Express each degree measure in radians. Leave your answer in terms of π.

1. 35° **2.** 252° **3.** 90° **4.** −30°

5. −60° **6.** 420° **7.** −135° **8.** 315°

9. 330° **10.** −150° **11.** 200° **12.** −306°

Example 2	Express in degrees: **a.** $\frac{7\pi}{6}$	**b.** $-\frac{3\pi}{2}$
Solution	**a.** $\frac{7\pi}{6}$ radians $= \frac{7\pi}{6} \cdot \frac{180°}{\pi} = 210°$	**b.** $-\frac{3\pi}{2}$ radians $= -\frac{3\pi}{2} \cdot \frac{180°}{\pi} = -270°$

Express each radian measure in degrees.

13. $\frac{11\pi}{12}$ **14.** $\frac{19\pi}{15}$ **15.** $\frac{\pi}{4}$ **16.** $-\frac{13\pi}{6}$

17. $-\frac{4\pi}{3}$ **18.** $-\frac{2\pi}{3}$ **19.** 4π **20.** $\frac{17\pi}{12}$

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13–1 Radian Measure *(continued)*

Express each radian measure in degrees. Leave your answer in terms of π.

21. 5 **22.** 2 **23.** -3 **24.** $\frac{1}{2}$

> ***Example 3*** a. Express 48° in radians to the nearest hundredth of a radian.
> b. Express 2.3 radians in degrees to the nearest tenth of a degree.
>
> ***Solution*** a. $48° = 48 \cdot \frac{\pi}{180}$ radians
> $\approx \frac{48(3.1416)}{180}$ radians
> ≈ 0.84 radians
>
> b. $2.3 \text{ radians} = 2.3 \cdot \frac{180°}{\pi}$
> $\approx \frac{2.3(180°)}{3.1416}$
> $\approx 131.8°$

Express each degree measure in radians. Give answers to the nearest hundredth of a radian.

25. 15° **26.** $-70°$ **27.** 83° **28.** 280°

29. 54° **30.** 312° **31.** $-162°$ **32.** 221°

Express each radian measure in degrees. Give answers to the nearest tenth of a degree.

33. 2.6 **34.** 4.5 **35.** 0.6 **36.** -1.3

37. 4 **38.** -6 **39.** 3.1 **40.** 7

> ***Example 4*** A central angle of a circle of radius 2 cm measures 1.3 radians. Find (a) the length of the intercepted arc and (b) the area of the related sector.
>
> ***Solution*** a. Use the formula $s = r\theta$.
> $s = 2(1.3)$
> $= 2.6$ cm
>
> b. Use the formula $A = \frac{1}{2}r^2\theta$.
> $A = \frac{1}{2}(2^2)(1.3)$
> $= 2.6 \text{ cm}^2$

A central angle of a circle of radius r measures θ radians. For the given values of r and θ, find (a) the length of the intercepted arc and (b) the area of the related sector.

41. $r=4$, $\theta=2$ **42.** $r=5$, $\theta=0.5$ **43.** $r=6$, $\theta=3$ **44.** $r=10$, $\theta=2.5$

Mixed Review Exercises

Solve each triangle. Give lengths to three significant digits and angle measures to the nearest tenth of a degree.

1. $a=10$, $b=6$, $\angle C = 53°$ **2.** $\angle A=80°$, $\angle B=35°$, $b=10$

3. $b=12$, $c=8$, $\angle B=110°$ **4.** $a=4$, $b=5$, $c=7$

5–8. Find the area of each triangle in Exercises 1–4. Give answers to three significant digits.

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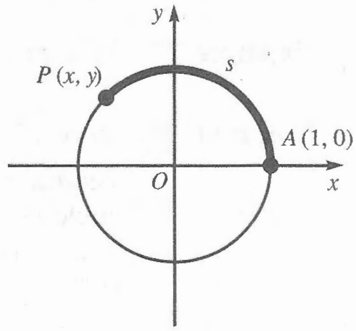
13–2 Circular Functions

Objective: To define the circular functions.

Vocabulary

Circular functions Let O be the unit circle $x^2 + y^2 = 1$, and let A be the point (1, 0). Given any real number s, start at A and measure $|s|$ units around O in a counterclockwise direction if $s \geq 0$ and in a clockwise direction if $s < 0$, arriving at a point $P(x, y)$. Then the circular functions of s are defined as follows:



$\sin s = y$ $\cos s = x$

$\tan s = \dfrac{\sin s}{\cos s}$ if $\cos s \neq 0$ $\cot s = \dfrac{\cos s}{\sin s}$ if $\sin s \neq 0$

$\sec s = \dfrac{1}{\cos s}$ if $\cos s \neq 0$ $\csc s = \dfrac{1}{\sin s}$ if $\sin s \neq 0$

Note: A circular function of a number s is equal to the corresponding trigonometric function of any angle having radian measure s. Therefore, the circular functions can be referred to as trigonometric functions; the context should make it clear whether functions of angles or functions of numbers are intended.

Example 1 For some number s, the point $\left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ is s units from $A(1, 0)$ along a unit circle. Find the exact values of the six circular functions of s.

Solution Use the definitions of the circular functions given above.

$\sin s = \dfrac{2\sqrt{2}}{3}$ $\cos s = -\dfrac{1}{3}$ $\tan s = \dfrac{\frac{2\sqrt{2}}{3}}{-\frac{1}{3}} = -2\sqrt{2}$

$\csc s = \dfrac{1}{\frac{2\sqrt{2}}{3}} = \dfrac{3}{2\sqrt{2}} = \dfrac{3\sqrt{2}}{4}$ $\sec s = \dfrac{1}{-\frac{1}{3}} = -3$ $\cot s = \dfrac{-\frac{1}{3}}{\frac{2\sqrt{2}}{3}} = -\dfrac{1}{2\sqrt{2}} = -\dfrac{\sqrt{2}}{4}$

For some number s, the point P is s units from $A(1, 0)$ along the unit circle $x^2 + y^2 = 1$. Find the exact values of the six circular functions of s.

1. $P\left(-\frac{3}{5}, \frac{4}{5}\right)$ **2.** $P\left(\frac{5}{13}, -\frac{12}{13}\right)$ **3.** $P\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ **4.** $P\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$

Example 2 Find sin 1.35 to four significant digits.

Solution 1 **Using a Calculator** Set the calculator in radian mode.
$\sin 1.35 = 0.9757$ to four significant digits.

Solution 2 **Using Tables** Use Table 6 to find the sine of an angle having radian measure 1.35.
$\sin 1.35 = 0.9757$

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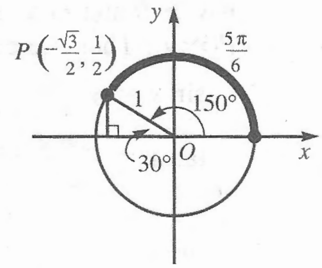
13–2 Circular Functions *(continued)*

Find the sine, cosine, and tangent of each number to four significant digits.

5. 1.17 **6.** 1.32 **7.** 0.41 **8.** 0.8

Example 3 Find the exact values of the six trigonometric functions of $\frac{5\pi}{6}$.

Solution Since $\frac{5\pi}{6}$ radians $= 150°$, use a $30° - 60° - 90°$ triangle. Because the hypotenuse of the triangle is a radius of the circle $x^2 + y^2 = 1$, the length of the hypotenuse is 1 unit. Recall that the lengths of the legs opposite the 30° and 60° angles in a $30° - 60° - 90°$ triangle are, respectively, $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse.



$\sin \frac{5\pi}{6} = \frac{1}{2}$ $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ $\tan \frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$

$\csc \frac{5\pi}{6} = 2$ $\sec \frac{5\pi}{6} = -\frac{2\sqrt{3}}{3}$ $\cot \frac{5\pi}{6} = -\sqrt{3}$

Find the exact values of the six trigonometric functions of the given number. If a function is undefined for the number, say so.

9. $\frac{2\pi}{3}$ **10.** $\frac{\pi}{2}$ **11.** $-\frac{\pi}{6}$ **12.** $\frac{7\pi}{6}$

13. 2π **14.** $-\frac{\pi}{4}$ **15.** $\frac{5\pi}{4}$ **16.** $\frac{5\pi}{3}$

Example 4 Find x to two decimal places if $\csc x = 1.077$ and $0 \le x \le \frac{\pi}{2}$.

Solution 1 **Using a Calculator**

Since $\csc x = 1.077$, $\sin x = \frac{1}{1.077} = 0.928505106$. Set the calculator in radian mode and use the inverse function key $\sin^{-1}$ (or inv sin).

$$x = \sin^{-1} 0.928505106 = 1.190366469$$

$\therefore x = 1.19$ to two decimal places.

Solution 2 **Using Tables**

Use Table 6 and read across 1.077 in the $\csc \theta$ column. You should find that $\csc 1.1903 = 1.077$.

$\therefore x = 1.19$ to two decimal places.

Find the number x to two decimal places if $0 \le x \le \frac{\pi}{2}$ and x has the given function value.

17. $\sin x = 0.6518$ **18.** $\tan x = 1.185$ **19.** $\csc x = 1.196$ **20.** $\cos x = 0.3809$

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13–3 Periodicity and Symmetry

Objective: To use periodicity and symmetry in graphing functions.

Vocabulary

Periodic function A function f is periodic if, for some positive constant p,

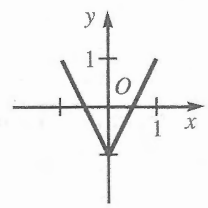
$$f(x + p) = f(x)$$

for every x in the domain of f. The smallest such p is the *period* of f.

Odd function A function f is odd if $f(-x) = -f(x)$ for every x in the domain of f. When a function f is odd, its graph is symmetric with respect to the origin. That is, if the point (x, y) is on the graph of f, then so is the point $(-x, -y)$.

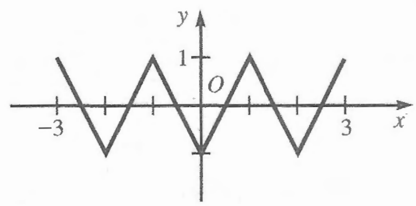
Even function A function f is even if $f(-x) = f(x)$ for every x in the domain of f. When a function f is even, its graph is symmetric with respect to the y-axis. That is, if the point (x, y) is on the graph of f, then so is the point $(-x, y)$.

Example 1 Part of the graph of a function f having period 2 is shown at the right. Graph f in the interval $-3 \le x \le 3$ and tell whether f is even, odd, or neither.

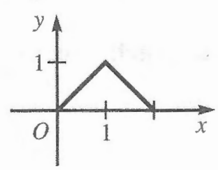


Solution Since the given part of the graph covers one period, repeat it to the right until you reach 3 and to the left until you reach -3.

The graph is symmetric with respect to the y-axis, so f is an even function.

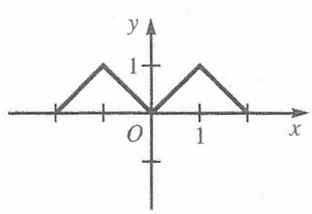


Example 2 Part of the graph of a function f is shown at the right. Graph f in the interval $-2 \le x \le 2$ assuming that f is (a) even and (b) odd.

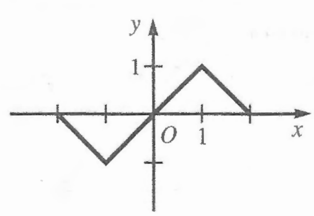


Solution

a. Reflect the given part of the graph in the y-axis.



b. Reflect the given part of the graph in the origin.



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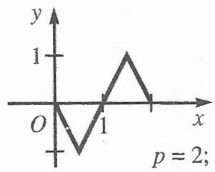
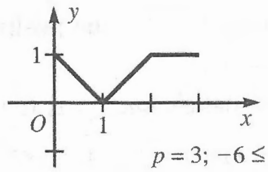
13–3 *Periodicity and Symmetry* (continued)

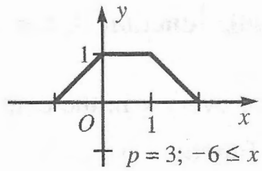
Part of the graph of a function *f* having the given period *p* is shown. Graph *f* in the indicated interval and tell whether *f* is even, odd, or neither.

1. 

$p = 2$; $-4 \le x \le 4$

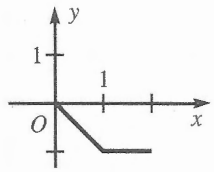
2. 

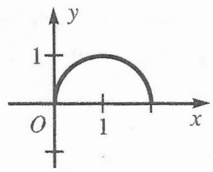
$p = 3$; $-6 \le x \le 6$

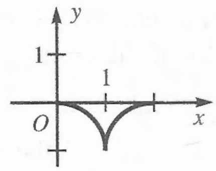
3. 

$p = 3$; $-6 \le x \le 6$

Part of the graph of a function *f* is given. Graph *f* in the interval $-2 \le x \le 2$ assuming *f* is (a) even and (b) odd.

4. 

5. 

6. 

Example 3 Determine whether each function is even, odd, or neither.

a. $f(x) = \dfrac{x^2}{x^2 + 2}$ **b.** $g(x) = x^3 - x$ **c.** $h(x) = x^2 + 2x$

Solution

a. $f(-x) = \dfrac{(-x)^2}{(-x)^2 + 2}$

$= \dfrac{x^2}{x^2 + 2} = f(x)$

$\therefore$ *f* is even.

b. $g(-x) = (-x)^3 - (-x)$

$= -x^3 + x = -g(x)$

$\therefore$ *g* is odd.

c. $h(-x) = (-x)^2 + 2(-x) = x^2 - 2x$

Notice that $x^2 - 2x \ne x^2 + 2x$ and $x^2 - 2x \ne -(x^2 + 2x)$.

$\therefore$ *h* is neither even nor odd.

Determine whether each function *f* is even, odd, or neither.

7. $f(x) = x^2 + 3x$ **8.** $f(x) = x^3 + 4x$ **9.** $f(x) = x^4 + x^2$

10. $f(x) = \dfrac{x^4}{x^2 + 2}$ **11.** $f(x) = x\sqrt[3]{x^3 + 1}$ **12.** $f(x) = x + \dfrac{1}{x}$

Mixed Review Exercises

Graph each equation.

1. $y = x^2 + 4x + 6$ **2.** $9x^2 + y^2 = 9$ **3.** $y = \log_2 x - 1$ **4.** $x^2 + y^2 + 2x = 3$

Solve.

5. $\sqrt{3x - 2} = x$ **6.** $\dfrac{2x}{x + 1} - \dfrac{1}{x} = \dfrac{2}{x^2 + x}$ **7.** $x^3 + 2x^2 + 2x - 5 = 0$

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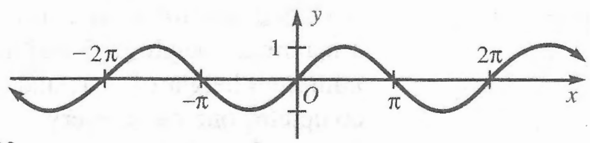
13–4 Graphs of the Sine and Cosine

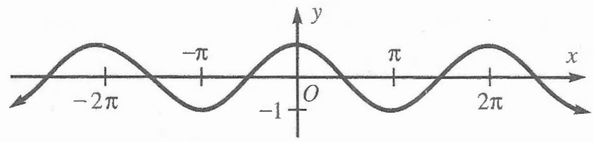
Objective: To graph the sine, cosine, and related functions.

Vocabulary

Graph of sine The graph of $y = \sin x$, called a *sine curve*, is shown at the right. Notice that the graph repeats itself every 2π units along the x-axis (because sine has period 2π), is symmetric with respect to the origin (because sine is an odd function), and has a maximum height of 1 and a minimum height of -1 (because the range of sine is $\{y: -1 \le y \le 1\}$).



Graph of cosine The graph of $y = \cos x$, called a *cosine curve*, is shown at the right. Notice that the graph repeats itself every 2π units along the x-axis (because cosine has period 2π), is symmetric with respect to the y-axis (because cosine is an even function), and has a maximum height of 1 and a minimum height of -1 (because the range of cosine is $\{y: -1 \le y \le 1\}$). Notice also that if the cosine curve were shifted $\frac{\pi}{2}$ units to the right, it would coincide with the sine curve.



Vertical shift The graph of a function of the form $y = c + \sin x$ (or $y = c + \cos x$) can be obtained from the graph of $y = \sin x$ (or $y = \cos x$) by shifting the graph upward c units if $c > 0$ or downward $|c|$ units if $c < 0$.

Amplitude For a function of the form $y = a \sin x$ (or $y = a \cos x$) where $a > 0$, the number a is called the amplitude of the function. Such a function has maximum value a and minimum value $-a$. That is, the constant a causes a vertical stretch or shrinking of the sine (or cosine) curve.

Maximum and minimum values The maximum value M and minimum value m of a function of the form $y = c + a \sin x$ (or $y = c + a \cos x$) where $a > 0$ are given by

$$M = c + a \qquad \text{and} \qquad m = c - a.$$

These equations can be rewritten to obtain the vertical shift and amplitude from M and m:

$$c = \frac{M + m}{2} \qquad \text{and} \qquad a = \frac{M - m}{2}.$$

Period The period of a function of the form $y = \sin bx$ (or $y = \cos bx$) where $b > 0$ is $\frac{2\pi}{b}$. The constant b causes a horizontal stretch or shrinking of the sine (or cosine) curve.

Example 1 For the function $y = 1 + 2 \sin \pi x$, find (a) the amplitude, (b) the maximum and minimum values, and (c) the period.

Solution The function has the form $y = c + a \sin bx$.

a. Amplitude: $a = 2$

b. Maximum value: $M = c + a = 1 + 2 = 3$
Minimum value: $m = c - a = 1 - 2 = -1$

c. Period: $\frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$

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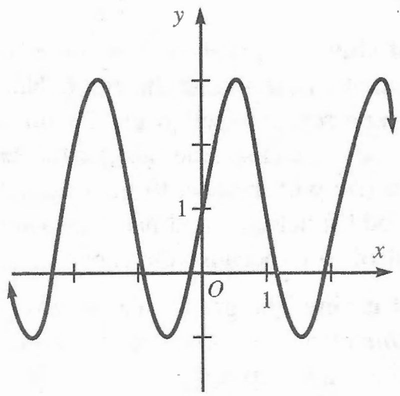
13-4 Graphs of the Sine and Cosine *(continued)*

Example 2 Graph the function given in Example 1. Show at least two periods.

Solution The graph is a sine curve that is shifted upward 1 unit, has a maximum height of 3 and a minimum height of -1, and completes one cycle every 2 units along the x-axis.



In Exercises 1–12:
a. Find the amplitude of each function.
b. Find the maximum and minimum values.
c. Find the period.

1. $y = 3 \sin x$ **2.** $y = \frac{1}{2}\cos x$ **3.** $y = \sin 2x$

4. $y = \cos 4x$ **5.** $y = 2 \sin 4x$ **6.** $y = \frac{1}{2}\cos 2x$

7. $y = \frac{1}{2}\sin 4x$ **8.** $y = 3 \cos \pi x$ **9.** $y = 2 \sin \frac{\pi}{2}x$

10. $y = 2 + 4 \cos \frac{\pi}{3}x$ **11.** $y = 3 \sin \pi x - 2$ **12.** $y = \frac{1}{2}\cos 2\pi x + 3$

13–16. Graph each function in Exercises 1, 4, 7, and 10. Show at least two periods.

Example 3 Find an equation of the form $y = c + a \sin bx$ that satisfies the following conditions: $M = 4$, $m = -2$, period 4π.

Solution

1. $a = \frac{M - m}{2} = \frac{4 - (-2)}{2} = 3$

2. $c = \frac{M + m}{2} = \frac{4 + (-2)}{2} = 1$

3. $\frac{2\pi}{b} = 4\pi$, so $b = \frac{2\pi}{4\pi} = \frac{1}{2}$

$\therefore y = 1 + 3 \sin \frac{1}{2}x$

Find an equation of the form $y = c + a \sin bx$ that satisfies the given conditions.

17. $M = 3$, $m = -1$, period π **18.** $M = 5$, $m = 1$, period 4 **19.** $M = 2$, $m = -2$, period 4π

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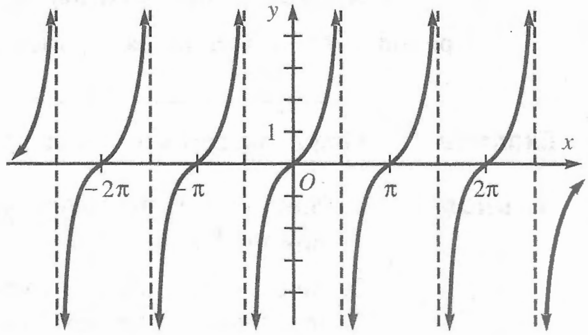
13–5 Graphs of the Other Functions

Objective: To graph the tangent, cotangent, secant, cosecant, and related functions.

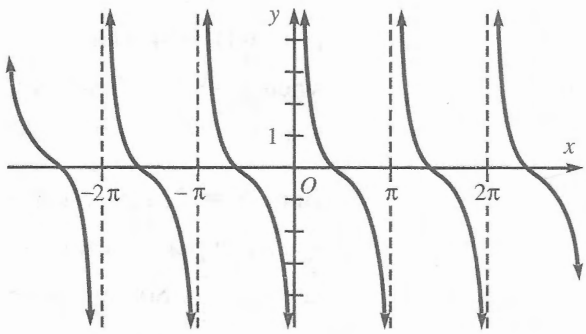
Vocabulary

Asymptote A line which a graph approaches without intersecting. Example: The lines $y = x$ and $y = -x$ are asymptotes for the graph of the hyperbola $x^2 - y^2 = 1$.

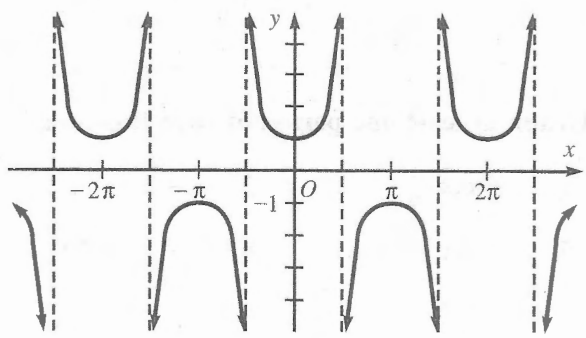
Graph of tangent The graph of $y = \tan x$ is shown at the right. Notice that the graph repeats itself every π units along the x-axis (because tangent has period π), is symmetric with respect to the origin (because tangent is an odd function), and has vertical asymptotes at $x = \frac{(2k + 1)\pi}{2}$ where k is an integer (because tangent is undefined for odd multiples of $\frac{\pi}{2}$).



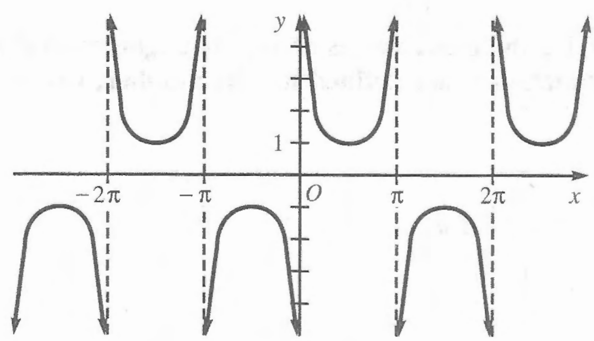
Graph of cotangent The graph of $y = \cot x$ is shown at the right. Notice that the graph repeats itself every π units along the x-axis (because cotangent has period π), is symmetric with respect to the origin (because cotangent is an odd function), and has vertical asymptotes at $x = k\pi$ where k is an integer (because cotangent is undefined for multiples of π).



Graph of secant The graph of $y = \sec x$ is shown at the right. Notice that the graph repeats itself every 2π units along the x-axis (because secant has period 2π), is symmetric with respect to the y-axis (because secant is an even function), and has vertical asymptotes at $x = \frac{(2k + 1)\pi}{2}$ where k is an integer (because secant is undefined for odd multiples of $\frac{\pi}{2}$).



Graph of cosecant The graph of $y = \csc x$ is shown at the right. Notice that the graph repeats itself every 2π units along the x-axis (because cosecant has period 2π), is symmetric with respect to the origin (because cosecant is an odd function), and has vertical asymptotes at $x = k\pi$ where k is an integer (because cosecant is undefined for multiples of π).



13–5 Graphs of the Other Functions *(continued)*

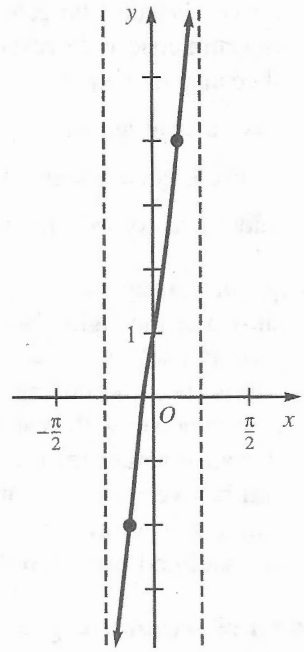
Effect of constants on the graphs of the trigonometric functions For an equation of the form $y = c + a[f(bx)]$ where f is one of the six trigonometric functions, the constants c, a, and b ($a, b > 0$) have the following effects on the graph of f:

1. c produces a vertical shift.
2. a produces a vertical stretching or shrinking.
3. b produces a horizontal stretching or shrinking. (Specifically, if the period of f is p, then the new period is $\frac{p}{b}$.)

Example Graph one period of the function $y = 1 + 3\tan 2x$.

Solution

1. Since $c = 1$, the tangent curve is shifted upward 1 unit.
2. Since $a = 3$, there is a vertical stretch of the tangent curve. For example, since $\tan x = 1$ when $x = \frac{\pi}{4}$, $1 + 3\tan 2x = 1 + 3(1) = 4$ when $2x = \frac{\pi}{4}$ $\left(\text{that is, when } x = \frac{\pi}{8}\right)$. Similarly, $1 + 3\tan 2x = 1 + 3(-1) = -2$ when $x = -\frac{\pi}{8}$.
3. Since $b = 2$, the period of the function is $\frac{\pi}{2}$. Asymptotes occur when $2x = \frac{\pi}{2}$ and $2x = -\frac{\pi}{2}$. Solving these equations gives asymptotes at $x = \frac{\pi}{4}$ and $x = -\frac{\pi}{4}$.



Graph at least one period of each function.

1. $y = \cot \frac{1}{2}x$
2. $y = 3\sec x$
3. $y = \frac{3}{2}\tan x + 2$
4. $y = \frac{1}{2}\csc \frac{\pi}{2}x$
5. $y = \sec \pi x - 3$
6. $y = 2\tan 2x$
7. $y = \csc 2x + 1$
8. $y = 3\cot x - 2$

Mixed Review Exercises

Give the exact values of the six trigonometric functions of each number. If a function is not defined for the number, say so.

1. $\frac{7\pi}{2}$
2. $-\frac{2\pi}{3}$
3. $\frac{11\pi}{6}$
4. $-\frac{7\pi}{4}$

Evaluate if $a = -3$ and $b = 9$.

5. $\frac{a + b}{a - b}$
6. $\log_b(-a)$
7. $\sum_{n=1}^{4} b \cdot a^{n-1}$
8. $b^{-a/2}$

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13–6 The Fundamental Identities

Objective: To simplify trigonometric expressions and to prove identities.

Vocabulary

The Reciprocal Identities

$\sin \alpha = \frac{1}{\csc \alpha}$ $\sin \alpha \csc \alpha = 1$ $\csc \alpha = \frac{1}{\sin \alpha}$

$\cos \alpha = \frac{1}{\sec \alpha}$ $\cos \alpha \sec \alpha = 1$ $\sec \alpha = \frac{1}{\cos \alpha}$

$\tan \alpha = \frac{1}{\cot \alpha}$ $\tan \alpha \cot \alpha = 1$ $\cot \alpha = \frac{1}{\tan \alpha}$

$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$ $\cot \alpha = \frac{\cos \alpha}{\sin \alpha}$

The Cofunction Identities

$\sin \theta = \cos (90° - \theta)$ $\cos \theta = \sin (90° - \theta)$ $\tan \theta = \cot (90° - \theta)$

$\csc \theta = \sec (90° - \theta)$ $\sec \theta = \csc (90° - \theta)$ $\cot \theta = \tan (90° - \theta)$

The Pythagorean Identities

$\sin^2 \alpha + \cos^2 \alpha = 1$ $1 + \tan^2 \alpha = \sec^2 \alpha$ $1 + \cot^2 \alpha = \csc^2 \alpha$

General strategies for proving identities You can't prove an identity by "working across the = sign" (for example, by cross multiplying), because doing so assumes that what is being proved is true. Instead, you can usually use one of the following strategies.

1. Simplify the more complicated side of the identity until it is identical to the other side.
2. Transform both sides of the identity into the same expression.

Special strategies for proving identities There are often several ways of proving identities. Here are some special strategies that can be helpful.

1. Express functions in terms of sine and cosine.
2. Look for expressions to which the Pythagorean identities can be applied. Example: $\cot^2 x - \csc^2 x = -1$
3. Use factoring. Example: $1 - \sin^2 x = (1 + \sin x)(1 - \sin x)$
4. Combine terms on each side of the identity into a single fraction.
5. Multiply one side of the identity by an expression equal to 1.

Example 1 Simplify: **a.** $\csc \theta \tan \theta$ **b.** $\frac{1 - \sin^2 x}{\sin x \cos x}$

Solution **a.** Write each function in terms of sine and cosine. Then simplify.

$$\csc \theta \tan \theta = \frac{1}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta} = \frac{1}{\cos \theta} = \sec \theta$$

(Solution continues on the next page.)

13-6 The Fundamental Identities *(continued)*

b. From the first Pythagorean identity you have $1 - \sin^2 x = \cos^2 x$.

$$\frac{1 - \sin^2 x}{\sin x \cos x} = \frac{\cos^2 x}{\sin x \cos x} = \frac{\cos x}{\sin x} = \cot x$$

Simplify.

1. $\sin \alpha \sec \alpha \cot \alpha$
2. $\cos^2 45° + \sin^2 45°$
3. $(\csc x - 1)(\csc x + 1)$
4. $\dfrac{\cot x}{\cos x}$
5. $\csc \alpha (\sec \alpha - \cos \alpha)$
6. $\dfrac{1 + \cot^2 \theta}{1 + \tan^2 \theta}$
7. $\dfrac{\cos \alpha}{\csc \alpha} - \dfrac{\sin \alpha}{\sec \alpha}$
8. $\dfrac{\csc x - \sin x}{\cos x}$
9. $\cot^2 (90° - \theta) - \sec^2 \theta$

Example 2 Write $\dfrac{\tan^2 x}{1 + \sec x}$ in terms of $\cos x$.

Solution From the second Pythagorean identity you have $\tan^2 x = \sec^2 x - 1$.

$$\frac{\tan^2 x}{1 + \sec x} = \frac{\sec^2 x - 1}{1 + \sec x} = \frac{(\sec x + 1)(\sec x - 1)}{1 + \sec x}$$

$$= \sec x - 1 = \frac{1}{\cos x} - 1 = \frac{1 - \cos x}{\cos x}$$

Write the first function in terms of the second.

10. $\cos \alpha \cot \alpha$; $\sin \alpha$
11. $\csc^2 x - 1$; $\cos x$
12. $\dfrac{\cot^2 \theta}{1 - \sin^2 \theta}$; $\cos \theta$

Example 3 Prove that $\dfrac{\cos x}{1 + \sin x} = \dfrac{1 - \sin x}{\cos x}$.

Solution Although each side is as "complicated" as the other, work only with the left side.

$$\frac{\cos x}{1 + \sin x} \stackrel{?}{=} \frac{1 - \sin x}{\cos x}$$

$$\frac{1 - \sin x}{1 - \sin x} \cdot \frac{\cos x}{1 + \sin x}$$ Use special strategy 5.

$$\frac{(1 - \sin x) \cos x}{1 - \sin^2 x}$$

$$\frac{(1 - \sin x) \cos x}{\cos^2 x}$$ Use special strategy 2.

$$\frac{1 - \sin x}{\cos x} = \frac{1 - \sin x}{\cos x}$$

Prove each identity.

13. $\tan x (\cot x + \tan x) = \sec^2 x$
14. $1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1$
15. $\sec \theta - \cos \theta = \sin \theta \tan \theta$
16. $\sec^4 \alpha - \tan^4 \alpha = 2 \sec^2 \alpha - 1$

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13–7 Trigonometric Addition Formulas

Objective: To use formulas for the sine and cosine of a sum or difference.

Vocabulary

Addition Formulas for the Sine and Cosine

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\quad$ $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\quad$ $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

CAUTION $\sin(\alpha + \beta) \neq \sin\alpha + \sin\beta$. Here is a counterexample:

$\sin(30° + 60°) = \sin 90° = 1$ but $\sin 30° + \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1 + \sqrt{3}}{2}$

Also, $\cos(\alpha + \beta) \neq \cos\alpha + \cos\beta$; $\sin(\alpha - \beta) \neq \sin\alpha - \sin\beta$; and $\cos(\alpha - \beta) \neq \cos\alpha - \cos\beta$.

Example 1 Find the exact value: **a.** $\cos 165°$ **b.** $\sin 195°$

Solution **a.**
$$\begin{aligned}\cos 165° &= \cos(120° + 45°)\\ &= \cos 120°\cos 45° - \sin 120°\sin 45°\\ &= \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\\ &= \frac{-\sqrt{2} - \sqrt{6}}{4}, \text{ or } -\frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

Note: There are often several ways to evaluate an expression like $\cos 165°$, such as $\cos(120° + 45°)$, $\cos(210° - 45°)$, and $\cos(135° + 30°)$. You should get the same value in each case.

b.
$$\begin{aligned}\sin 195° &= \sin(225° - 30°)\\ &= \sin 225°\cos 30° - \cos 225°\sin 30°\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

Find the exact value of each of the following.

1. $\sin 105°$ **2.** $\cos 15°$ **3.** $\cos 255°$

4. $\sin 285°$ **5.** $\sin 345°$ **6.** $\cos 75°$

Example 2 Simplify to a trigonometric function of a single angle. Then give the exact value if possible.

a. $\cos 50°\cos 20° + \sin 50°\sin 20°$ **b.** $\sin 70°\cos 20° + \cos 70°\sin 20°$

Solution **a.**
$$\begin{aligned}\cos 50°\cos 20° + \sin 50°\sin 20° &= \cos(50° - 20°)\\ &= \cos 30° = \frac{\sqrt{3}}{2}\end{aligned}$$

(Solution continues on the next page.)

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13–7 *Trigonometric Addition Formulas* *(continued)*

b. $\sin 70° \cos 20° + \cos 70° \sin 20° = \sin (70° + 20°)$
$= \sin 90° = 1$

Simplify to a trigonometric function of a single angle. Then give the exact value if possible.

7. $\sin 65° \cos 5° - \cos 65° \sin 5°$

8. $\sin 75° \cos 15° + \cos 75° \sin 15°$

9. $\cos 25° \cos 20° - \sin 25° \sin 20°$

10. $\cos 110° \cos 50° + \sin 110° \sin 50°$

11. $\sin \frac{\pi}{6} \cos \frac{\pi}{12} + \cos \frac{\pi}{6} \sin \frac{\pi}{12}$

12. $\cos \frac{5\pi}{12} \cos \frac{\pi}{12} - \sin \frac{5\pi}{12} \sin \frac{\pi}{12}$

13. $\cos 2\alpha \cos 3\alpha + \sin 2\alpha \sin 3\alpha$

14. $\sin 5\theta \cos \theta - \cos 5\theta \sin \theta$

Example 3 Prove the identity $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

Solution

$$\cos\left(\frac{\pi}{2} - x\right) \stackrel{?}{=} \sin x$$
$$\cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x$$
$$0 \cdot \cos x + 1 \cdot \sin x$$
$$\sin x = \sin x$$

Prove each identity.

15. $\cos (x - \pi) = -\cos x$

16. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$

17. $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos \theta$

18. $\sin (\pi - x) + \sin (\pi + x) = 0$

19. $\cos\left(\frac{3\pi}{2} - x\right) + \cos\left(\frac{3\pi}{2} + x\right) = 0$

20. $\cos\left(\frac{\pi}{3} + \theta\right) + \cos\left(\frac{\pi}{3} - \theta\right) = \cos \theta$

Mixed Review Exercises

Graph each function. Show at least two periods.

1. $y = 2 \cos \frac{1}{2}x$

2. $y = \tan \frac{1}{2}x$

3. $y = 1 + \sin \pi x$

Simplify.

4. $\cos^2 x(1 + \tan^2 x)$

5. $x^8(x^{\pi-2})^4$

6. $\sqrt[5]{64x^{12}}$

7. $\log_3 \frac{1}{27}$

8. $\frac{3 + \sqrt{3}}{1 + \sqrt{3}}$

9. $\sqrt{\frac{1 + \frac{3}{5}}{2}}$

10. $\left(\frac{x^{3/4}}{x^{1/4}}\right)^8$

11. $\frac{2x}{x^2 - 1} - \frac{1}{x - 1}$

12. $\sec x - \sin x \tan x$

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13–8 Double-Angle and Half-Angle Formulas

Objective: To use the double-angle and half-angle formulas for the sine and cosine.

Vocabulary

Double-Angle Formulas for Sine and Cosine

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \qquad \cos 2\theta = 1 - 2 \sin^2 \theta \qquad \cos 2\theta = 2 \cos^2 \theta - 1$$

Half-Angle Formulas for Sine and Cosine

$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}} \qquad \cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$$

Note: The quadrant in which $\frac{\theta}{2}$ is located determines the sign (+ or −) you use.

Example 1 Simplify to a trigonometric function of a single angle. Do not evaluate.

a. $1 - 2 \sin^2 20°$ **b.** $\sqrt{\frac{1 - \cos 110°}{2}}$

Solution

a. $1 - 2 \sin^2 20° = \cos 2(20°)$
$= \cos 40°$

b. $\sqrt{\frac{1 - \cos 110°}{2}} = \sin \frac{110°}{2}$
$= \sin 55°$

Example 2 Find the exact value: **a.** $\sin \frac{3\pi}{8}$ **b.** $\cos 105°$

Solution

a. Since $\frac{3\pi}{8}$ is a first-quadrant angle, $\sin \frac{3\pi}{8}$ is positive.

$$\sin \frac{3\pi}{8} = \sin \frac{1}{2}\left(\frac{3\pi}{4}\right)$$
$$= +\sqrt{\frac{1 - \cos\frac{3\pi}{4}}{2}}$$
$$= \sqrt{\frac{1 - \left(-\frac{\sqrt{2}}{2}\right)}{2}}$$
$$= \sqrt{\frac{2 + \sqrt{2}}{4}}$$
$$= \frac{1}{2}\sqrt{2 + \sqrt{2}}$$

b. Since 105° is a second-quadrant angle, cos 105° is negative.

$$\cos 105° = \cos \frac{1}{2}(210°)$$
$$= -\sqrt{\frac{1 + \cos 210°}{2}}$$
$$= -\sqrt{\frac{1 + \left(-\frac{\sqrt{3}}{2}\right)}{2}}$$
$$= -\sqrt{\frac{2 - \sqrt{3}}{4}}$$
$$= -\frac{1}{2}\sqrt{2 - \sqrt{3}}$$

Simplify to a trigonometric function of a single angle. Do not evaluate.

1. $2 \sin 20° \cos 20°$
2. $\cos^2 25° - \sin^2 25°$
3. $1 - 2 \sin^2 65°$
4. $2 \cos^2 35° - 1$
5. $\sqrt{\frac{1 - \cos 70°}{2}}$
6. $\sqrt{\frac{1 + \cos 50°}{2}}$

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13–8 Double-Angle and Half-Angle Formulas *(continued)*

Simplify to a trigonometric function of a single angle. Do not evaluate.

7. $\sqrt{\dfrac{1 - \cos 62^\circ}{2}}$ **8.** $\sqrt{\dfrac{1 + \cos 40^\circ}{2}}$ **9.** $2 \sin 4t \cos 4t$

10. $\cos^2 3t - \sin^2 3t$ **11.** $1 - 2 \sin^2 \dfrac{1}{2}\theta$ **12.** $2 \cos^2 2\theta - 1$

Use a half-angle or double-angle formula to find the exact value of each of the following.

13. $\cos^2 15^\circ - \sin^2 15^\circ$ **14.** $1 - 2 \sin^2 22.5^\circ$ **15.** $2 \sin 75^\circ \cos 75^\circ$

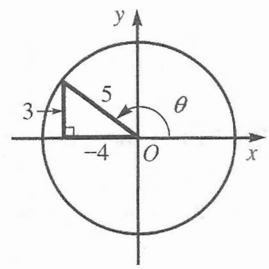
16. $2 \sin \dfrac{\pi}{12} \cos \dfrac{\pi}{12}$ **17.** $2 \cos^2 165^\circ - 1$ **18.** $1 - 2 \sin^2 \dfrac{\pi}{8}$

19. $\cos 67.5^\circ$ **20.** $\cos 75^\circ$ **21.** $\sin 22.5^\circ$

22. $\sin \dfrac{5\pi}{8}$ **23.** $\cos \dfrac{11\pi}{12}$ **24.** $\sin 75^\circ$

Example 3 If $\cos \theta = -\dfrac{4}{5}$ and $0^\circ < \theta < 180^\circ$, find (a) $\sin 2\theta$ and (b) $\cos \dfrac{\theta}{2}$.

Solution If $0^\circ < \theta < 180^\circ$, θ is either a first- or a second-quadrant angle. Since $\cos \theta$ is negative, θ must be a second-quadrant angle. You can find the exact value of $\sin \theta$ in one of two ways:



1. Make a sketch of θ as shown at the right. You can see from the sketch that if $\cos \theta = -\dfrac{4}{5}$, then $\sin \theta = \dfrac{3}{5}$.
2. In the second quadrant, $\sin \theta > 0$.

$$\therefore \sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \left(-\frac{4}{5}\right)^2} = \frac{3}{5}$$

a. $\sin 2\theta = 2 \sin \theta \cos \theta = 2\left(\dfrac{3}{5}\right)\left(-\dfrac{4}{5}\right) = -\dfrac{24}{25}$

b. Since $0^\circ < \theta < 180^\circ$, $0^\circ < \dfrac{\theta}{2} < 90^\circ$. Therefore $\cos \dfrac{\theta}{2}$ is positive.

$$\cos \frac{\theta}{2} = +\sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{1 + \left(-\frac{4}{5}\right)}{2}} = \sqrt{\frac{1}{10}} = \frac{\sqrt{10}}{10}$$

In Exercises 25–28, $0^\circ < \theta < 180^\circ$ and $\cos \theta = -\dfrac{5}{13}$. Find the following.

25. $\cos 2\theta$ **26.** $\sin 2\theta$ **27.** $\cos \dfrac{\theta}{2}$ **28.** $\sin \dfrac{\theta}{2}$

In Exercises 29–32, $180^\circ < \theta < 360^\circ$ and $\cos \theta = \dfrac{12}{13}$. Find the following.

29. $\cos 2\theta$ **30.** $\sin 2\theta$ **31.** $\cos \dfrac{\theta}{2}$ **32.** $\sin \dfrac{\theta}{2}$

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13–9 Formulas for the Tangent

Objective: To use the addition, double-angle, and half-angle formulas for the tangent.

Vocabulary

Addition Formulas for the Tangent

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Double-Angle Formula for the Tangent

$$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}$$

Half-Angle Formulas for the Tangent

$$\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad \tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta} \qquad \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}$$

Example 1 Simplify to a trigonometric function of a single angle. Do not evaluate.

a. $\dfrac{\tan 27^\circ+\tan 31^\circ}{1-\tan 27^\circ\tan 31^\circ}$ **b.** $\dfrac{1-\cos 43^\circ}{\sin 43^\circ}$

Solution **a.** $\dfrac{\tan 27^\circ+\tan 31^\circ}{1-\tan 27^\circ\tan 31^\circ}=\tan(27^\circ+31^\circ)$
$=\tan 58^\circ$

b. $\dfrac{1-\cos 43^\circ}{\sin 43^\circ}=\tan\dfrac{43^\circ}{2}$
$=\tan 21.5^\circ$

Simplify to a trigonometric function of a single angle. Do not evaluate.

1. $\dfrac{\tan 52^\circ+\tan 37^\circ}{1-\tan 52^\circ\tan 37^\circ}$
2. $\dfrac{1-\cos 330^\circ}{\sin 330^\circ}$
3. $\dfrac{2\tan 35^\circ}{1-\tan^2 35^\circ}$
4. $\sqrt{\dfrac{1-\cos 42^\circ}{1+\cos 42^\circ}}$
5. $\dfrac{\tan 230^\circ-\tan 40^\circ}{1+\tan 230^\circ\tan 40^\circ}$
6. $\dfrac{\sin 110^\circ}{1+\cos 110^\circ}$
7. $\dfrac{\tan 76^\circ-\tan 22^\circ}{1+\tan 76^\circ\tan 22^\circ}$
8. $\dfrac{2\tan\pi}{1-\tan^2\pi}$
9. $\dfrac{\tan 3\pi+\tan\pi}{1-\tan 3\pi\tan\pi}$

Example 2 Find the exact value: **a.** $\tan 165^\circ$ **b.** $\tan\dfrac{3\pi}{8}$

Solution **a.** $\tan 165^\circ=\tan(120^\circ+45^\circ)=\dfrac{\tan 120^\circ+\tan 45^\circ}{1-\tan 120^\circ\tan 45^\circ}$

$$=\frac{-\sqrt{3}+1}{1-(-\sqrt{3})(1)}$$

$$=\frac{-\sqrt{3}+1}{1+\sqrt{3}}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$=\frac{4-2\sqrt{3}}{-2}=-2+\sqrt{3}$$

(*Solution continues on the next page.*)

13–9 Formulas for the Tangent *(continued)*

b. $\tan \frac{3\pi}{8} = \tan \frac{1}{2}\left(\frac{3\pi}{4}\right) = \dfrac{1 - \cos \frac{3\pi}{4}}{\sin \frac{3\pi}{4}}$

$$= \dfrac{1 - \left(-\frac{\sqrt{2}}{2}\right)}{\frac{\sqrt{2}}{2}} = \frac{2 + \sqrt{2}}{\sqrt{2}} = \sqrt{2} + 1$$

Find the exact value of each of the following.

10. $\tan 105°$ **11.** $\tan 247.5°$ **12.** $\tan 345°$ **13.** $\tan 195°$

14. $\tan \frac{7\pi}{8}$ **15.** $\tan \frac{19\pi}{12}$ **16.** $\tan 292.5°$ **17.** $\tan 375°$

Example 3 If $\tan \alpha = -\frac{3}{4}$, $\tan \beta = \frac{5}{12}$, and α is a second-quadrant angle, find:

a. $\tan (\alpha - \beta)$ **b.** $\tan 2\alpha$

Solution **a.** $\tan (\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

$$= \dfrac{-\frac{3}{4} - \frac{5}{12}}{1 + \left(-\frac{3}{4}\right)\left(\frac{5}{12}\right)}$$

$$= \dfrac{-\frac{14}{12}}{\frac{33}{48}} = -\frac{56}{33}$$

b. $\tan 2\alpha = \dfrac{2 \tan \alpha}{1 - \tan^2 \alpha}$

$$= \dfrac{2\left(-\frac{3}{4}\right)}{1 - \left(-\frac{3}{4}\right)^2}$$

$$= \dfrac{-\frac{3}{2}}{\frac{7}{16}} = -\frac{24}{7}$$

In Exercises 18–21, $\tan \alpha = -\frac{12}{5}$, $\tan \beta = \frac{4}{3}$, and α is a second-quadrant angle. Find the value of each expression.

18. $\tan (\alpha + \beta)$ **19.** $\tan (\alpha - \beta)$ **20.** $\tan 2\alpha$ **21.** $\tan \frac{\alpha}{2}$

Mixed Review Exercises

Prove each identity.

1. $\csc^2 \theta(1 - \sin^2 \theta) = \cot^2 \theta$ **2.** $\tan \theta(\sin \theta - \csc \theta) = -\cos \theta$

3. $(\csc x - \cot x)(\csc x + \cot x) = 1$ **4.** $\csc 2\alpha = \frac{1}{2}\sec \alpha \csc \alpha$

Solve each inequality and graph the solution set.

5. $|x + 1| > 2$ **6.** $x^2 < 4x - 3$ **7.** $3 \le 4x - 1 < 7$

8. $|3x + 2| \le 1$ **9.** $5 - 2x < -3$ **10.** $x^3 - 4x > 0$