**Math 122 - Test 1**
**February 18, 2019**

**Name** Key

**Grade:** $\frac{\quad}{80} =$ ____

*No calculators. Show your work. Points will be deducted for inadequate explanations, using mathematical notation incorrectly, illegibility, not following directions, and not making obvious and conventional simplifications.*

**1.** Differentiate $f(x) = 3\sec x \tan x + \sin(2x^5)$. Simplify. (8 pts.)

$$f'(x) = 3\frac{d}{dx}(\sec x \tan x) + \frac{d}{dx}\sin(2x^5)$$

$$= 3\left(\sec x \frac{d}{dx}\tan x + \tan x \frac{d}{dx}\sec x\right) + \cos(2x^5)\frac{d}{dx}(2x^5)$$

$$= 3(\sec x \cdot \sec^2 x + \tan x \sec x \tan x + 10x^4\cos(2x^5)$$

$$= 3\sec^3 x + 3\tan^2 x \sec x + 10x^4\cos(2x^5)$$

$$= 3\left(\frac{1}{\cos^3 x} + \frac{\sin^2 x}{\cos^3 x}\right) + 10x^4\cos(2x^5)$$

$$= 3\,\frac{1+\sin^2 x}{\cos^3 x} + 10x^4\cos(2x^5)$$

**2.** Differentiate $y = (x^3-1)^5 + \cos^{-1}\left(\frac{1}{x^2}\right)$ with respect to $x$. Simplify. (8 pts.)

$$y' = 5(x^3-1)^4\frac{d}{dx}(x^3-1) - \frac{1}{\sqrt{1-\left(\frac{1}{x^2}\right)^2}} \cdot \frac{d}{dx}x^{-2}$$

$$= 5(x^3-1)^4(3x^2) - \frac{1}{\sqrt{1-\frac{1}{x^4}}}(-2x^{-3})$$

$$= 15x^2(x^3-1)^4 + \frac{2}{x^3\sqrt{\frac{x^4-1}{x^4}}}$$

$$= 15x^2(x^3-1)^4 + \frac{2}{x\sqrt{x^4-1}}$$



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**3.** Find the derivative $y'$ by implicit differentiation for $\sin(xy) = 1 - \cos y$. (8 pts.)

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}(1-\cos y)$$

$$\Rightarrow \cos(xy)\frac{d}{dx}(xy) = (\sin y)\cdot y'$$

$$\Rightarrow \cos(xy)(xy'+y) = y'\sin y$$

$$\Rightarrow x\cos(xy)\,y' + y\cos(xy) = y'\sin y$$

$$\Rightarrow [x\cos(xy) - \sin y]\,y' = -y\cos(xy)$$

$$\Rightarrow y' = \frac{y\cos(xy)}{\sin y - x\cos(xy)}$$

**4.** Find the following indefinite integrals:

**(a)** (6 pts.)

$$\int\left(\frac{1}{2\sqrt{x}} - \frac{x^3+4}{x}\right)dx \quad =: I$$

$$I = \frac{1}{2}\int x^{-1/2}dx - \int\left(x^2+\frac{4}{x}\right)dx$$

$$= x^{\frac{1}{2}} - \left(\frac{x^3}{3} + 4\ln|x|\right) + C$$

$$\Rightarrow I = \sqrt{x} - \frac{1}{3}x^3 - 4\ln|x| + C$$

**(b)** (6 pts.)

$$\int\left(12t^5 - 3\sec^2 t + \frac{\pi}{2}\right)dt \quad =: J$$

$$J = 12\int t^5\,dt - 3\int \sec^2 t\,dt + \frac{\pi}{2}\int dt$$

$$= 12\left(\frac{t^6}{6}\right) - 3\tan t + \frac{\pi}{2}t + C$$

$$= 2t^6 - 3\tan t + \frac{\pi}{2}t + C$$

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**5.** Pancake butter is poured onto a pan to form a circular pancake whose area increases at the rate of 4 square centimeters per second. How fast is the radius increasing when the diameter of the pancake is 10 centimeters? (8pts)

The area $A$ of the pancake is $A = \pi r^2$ where $r$ is the radius.
Differentiating wrt $t$, we have

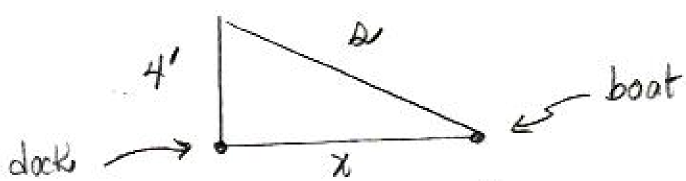
$$\frac{d}{dt} A = \frac{d}{dt} \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

Since $A' = 4 \frac{cm^2}{s}$, when $r = \frac{10cm}{2} = 5cm$, we have

$$4 = 2\pi(5)\frac{dr}{dt}.$$

$$\Rightarrow \frac{dr}{dt} = \frac{4}{10\pi} = \frac{0.4}{\pi} \frac{cm}{s}$$

**6.** A boat is pulled in toward a dock by a rope that is being wound up by a windlass at the rate of 3 feet per second. The windlass is 4 feet above the level of the water. How fast is the boat moving through the water when it is 12 feet from the dock? ( Include units in your answer.) (8 pts.)




Let $s$ = length of rope from windlass to boat (in feet).
Let $x$ = distance between boat and dock.

Given $\frac{ds}{dt} = -3 \frac{ft}{s}$

Since $s^2 = x^2 + 4^2$, we have

$$\frac{d}{dt} s^2 = \frac{d}{dt}(x^2 + 4^2) \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow x\frac{dx}{dt} = s\frac{ds}{dt}$$

When $x = 12$, then $12^2 + 4^2 = s^2 \Rightarrow s = \sqrt{144+16} = \sqrt{160} = \sqrt{16(10)} = 4\sqrt{10}$.

Thus $12\frac{dx}{dt} = 4\sqrt{10}(-3)$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{10} \text{ ft/s}$$

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**7.** Fill in the following blanks to complete the following statements: (12 pts.)

**(a)** The two hypotheses of the Mean Value Theorem are:

**(i)** a function $f$ is continuous on the closed interval $[a,b]$.

**(ii)** $f$ is differentiable on the open interval $(a,b)$.

The conclusion of the Mean Value Theorem is that there is a number $c$ in $(a,b)$ such that

$f(b) - f(a) = f'(c)(b-a)$.

**(b)** If $f'(x) = \sin x + \dfrac{2}{x}$ for all $x > 0$, then we know that $f(x) = -\cos x + 2\ln x + C$ where $C$ is an arbitrary constant.

**(c)** If $g$ and $h$ are two functions such that $\dfrac{d}{dx}g(x) = h(x)$, then $\int h(x)\,dx = g(x) + C$ where $C$ = arbitrary constant.

**8.** Find the formula for the position $s(t)$ of a particle if its acceleration $a(t)$ is

$$a(t) = 3t^2 - \frac{3}{2}\sqrt{t}$$

given that $s(0) = 0$, $s(1) = 2$. (8 pts.)

Let $v(t)$ denote the velocity of the particle.

Since $a(t) = \frac{d}{dt}v(t)$, we have

$$\frac{dv}{dt} = 3t^2 - \frac{3}{2}t^{1/2}.$$

Thus,

$$v(t) = \int \left(3t^2 - \frac{3}{2}t^{1/2}\right) dt = t^3 - t^{3/2} + C$$

Since $v(t) = \frac{ds}{dt}$,

$$\frac{ds}{dt} = t^3 - t^{3/2} + C.$$

And so

$$s(t) = \int \left(t^3 - t^{3/2} + C\right) dt = \frac{1}{4}t^4 - \frac{2}{5}t^{5/2} + Ct + K.$$

Since $s(0) = 0$, we have $\frac{1}{4}(0)^4 - \frac{2}{5}(0)^{5/2} + C(0) + K = 0 \Rightarrow K = 0.$

So,

$$s(t) = \frac{1}{4}t^4 - \frac{2}{5}t^{5/2} + Ct.$$

Letting $t = 1$ and $s(1) = 2$ yields

$$\frac{1}{4} - \frac{2}{5} + C = 2 \Rightarrow \frac{5-8}{20} + C = 2 \Rightarrow C = 2 + \frac{3}{20} = \frac{43}{20}.$$

$$\therefore\ s(t) = \frac{1}{4}t^4 - \frac{2}{5}t^{5/2} + \frac{43}{20}t$$

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**9.** Consider the region enclosed by the graph of $f(x) = x^2 + 2x$ and the $x$-axis from $x = 0$ to $x = 3$. Express the area of this region as the limit of a Riemann sum. Take the limit to find the area. (8 pts.)

Divide $[0, 3]$ into $n$ subintervals each of width $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$.

The endpoints of the intervals are $x_i = a + i\Delta x = 0 + i\left(\frac{3}{n}\right) \Rightarrow x_i = \frac{3i}{n}$, $i = 0, 1, \ldots, n$.

Riemann sum using right endpoints: Since $f(x) = x^2 + 2x$,

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} (x_i^2 + 2x_i)\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2} + \frac{6i}{n}\right).$$

$$\Rightarrow R_n = \frac{27}{n^3}\sum_{i=1}^{n} i^2 + \frac{18}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{18}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= \frac{9}{2}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 9\left(\frac{n+1}{n}\right)$$

$$= \frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 9\left(1+\frac{1}{n}\right).$$

$$\text{Area} = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 9\left(1+\frac{1}{n}\right)\right]$$

Taking the limit, $\text{Area} = \frac{9}{2}(1)(2) + 9(1) = 18$ unit$^2$.

**Extra Credit.** Express the integral $\int_{-1}^{5} \sqrt{1+x^3}\,dx$ as the limit of a Riemann sum. However, do not attempt to evaluate this limit. (4 pts.)

$f(x) = \sqrt{1+x^3}$, $\Delta x = \frac{b-a}{n} = \frac{5-(-1)}{n} = \frac{6}{n}$

$x_i = a + i\Delta x = -1 + i\left(\frac{6}{n}\right) = -1 + \frac{6i}{n}$

$$\int_{-1}^{5}\sqrt{1+x^3}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\sqrt{1+x_i^3}$$

$$\Rightarrow \int_{-1}^{3}\sqrt{1+x^3}\,dx = \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\sqrt{1+\left(-1+\frac{6i}{n}\right)^3}$$

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