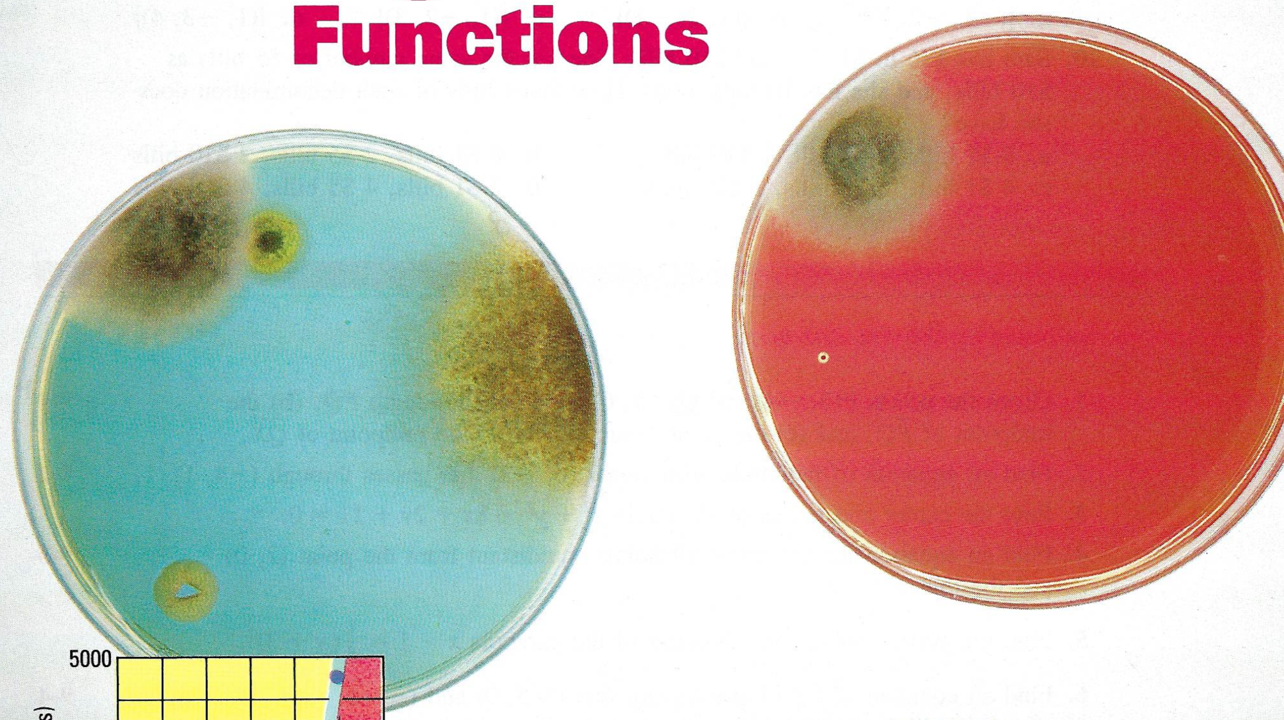
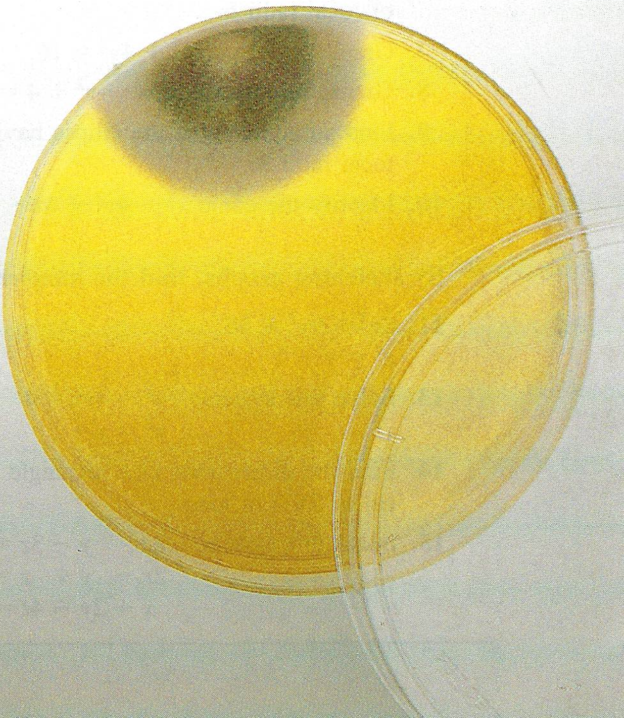
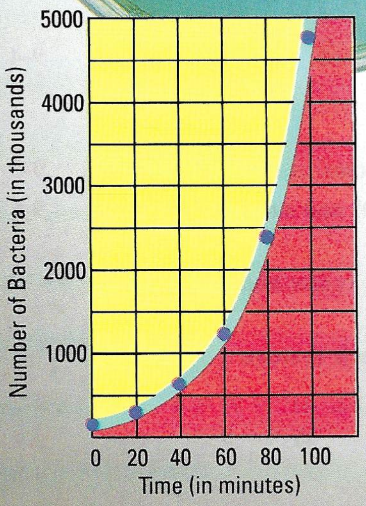
10 Exponential and Logarithmic Functions







In a Petri dish, a colony of bacteria can grow exponentially. The graph shows that a colony may double in size every 20 minutes.

Exponential Functions

10-1 Rational Exponents

Objective To extend the meaning of exponents to include rational numbers.

You know the meaning of the power b^x when x is an integer. For example,

$$5^2 = 25, \quad 5^{-2} = \frac{1}{25}, \quad \text{and} \quad 5^0 = 1.$$

Now you'll learn the meaning of b^x when x is any *rational number*.

Definition of $b^{1/2}$: (read as "b to the one-half power" or as "b to the one half") If the laws of exponents on page 212 are to hold, then

$$(b^{1/2})^2 = b^{(1/2)2} = b^1 = b.$$

Since the square of $b^{1/2}$ is b, $b^{1/2}$ *is defined to be* $\sqrt{b}$.

Definition of $b^{1/3}$: $b^{1/3}$ *is defined to be* $\sqrt[3]{b}$, since its cube is b:

$$(b^{1/3})^3 = b^{(1/3)3} = b^1 = b$$

Definition of $b^{2/3}$: Using the law $(a^m)^n = a^{mn}$, you can express $b^{2/3}$ in either of two ways:

$$b^{2/3} = (b^{1/3})^2 = (\sqrt[3]{b})^2 \quad \text{or} \quad b^{2/3} = (b^2)^{1/3} = \sqrt[3]{b^2}$$

Therefore, $b^{2/3}$ *is defined to be either of the equivalent expressions* $(\sqrt[3]{b})^2$ *or* $\sqrt[3]{b^2}$.

Definition of $b^{p/q}$

If p and q are integers with $q > 0$, and b is a positive real number, then

$$b^{p/q} = (\sqrt[q]{b})^p = \sqrt[q]{b^p}.$$

Example 1 Simplify $16^{3/4}$.

Solution 1 $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$ ***Answer***

Solution 2 $16^{3/4} = \sqrt[4]{16^3} = \sqrt[4]{4096} = 8$ ***Answer***

Notice that computing the root first is easier. This is usually the case.

Example 2 Simplify: **a.** $25^{-3/2}$ **b.** $9^{2.5}$

Solution

a. $25^{-3/2} = \dfrac{1}{25^{3/2}}$

$= \dfrac{1}{(\sqrt{25})^3}$

$= \dfrac{1}{125}$ ***Answer***

b. $9^{2.5} = 9^{5/2}$

$= (\sqrt{9})^5$

$= 3^5$

$= 243$ ***Answer***

The laws of exponents on page 212 also hold for powers with rational exponents. These laws can be used with the definition of rational exponents to simplify many radical expressions. The first step is to write the expression in **exponential form,** that is, as a power or a product of powers.

Example 3 **a.** Write $\sqrt[3]{\dfrac{a^5b^3}{c^2}}$ in exponential form. **b.** Simplify $\left(\dfrac{1}{\sqrt[3]{4}}\right)^{-3/2}$.

Solution

a. $\sqrt[3]{\dfrac{a^5b^3}{c^2}} = \left(\dfrac{a^5b^3}{c^2}\right)^{1/3}$

$= \dfrac{a^{5/3}b^{3/3}}{c^{2/3}}$

$= a^{5/3}bc^{-2/3}$ ***Answer***

b. $\left(\dfrac{1}{\sqrt[3]{4}}\right)^{-3/2} = (4^{-1/3})^{-3/2}$

$= 4^{(-1/3)(-3/2)}$

$= 4^{1/2}$

$= 2$ ***Answer***

Example 4 Write $\sqrt{8} \cdot \sqrt[3]{4}$ in exponential form and in simplest radical form.

Solution

$\sqrt{8} \cdot \sqrt[3]{4} = \sqrt{2^3} \cdot \sqrt[3]{2^2}$

$= 2^{3/2} \cdot 2^{2/3}$

$= 2^{13/6}$ ⟵ in exponential form ***Answer***

$= 2^2 \cdot 2^{1/6}$

$= 4\sqrt[6]{2}$ ⟵ in simplest radical form ***Answer***

Example 5 Solve: **a.** $(x - 1)^{3/2} = 8$ **b.** $5x^{-1/3} = 20$

Solution

a. $(x - 1)^{3/2} = 8$

$[(x - 1)^{3/2}]^{2/3} = 8^{2/3}$ (Raise both sides to the $\frac{2}{3}$ power.)

$x - 1 = 4$

$x = 5$

$\therefore$ the solution set is $\{5\}$.

Check: $(5 - 1)^{3/2} = 4^{3/2} = 8$ ✓

b. $5x^{-1/3} = 20$

$x^{-1/3} = 4$

$(x^{-1/3})^{-3} = 4^{-3}$ (Raise both sides to the -3 power.)

$x = \dfrac{1}{64}$

$\therefore$ the solution set is $\left\{\dfrac{1}{64}\right\}$.

Check: $5\left(\dfrac{1}{64}\right)^{-1/3} = 5 \cdot 4 = 20$ ✓

Oral Exercises

Simplify.

1. a. $9^{1/2}$ **b.** $9^{-1/2}$

2. a. $8^{1/3}$ **b.** $8^{-1/3}$

3. a. $8^{2/3}$ **b.** $8^{-2/3}$

4. a. $16^{1/4}$ **b.** $16^{-1/4}$

5. $27^{-1/3}$

6. $27^{2/3}$

7. $4^{3/2}$

8. $25^{-1/2}$

Give the letter of the correct answer. Assume *x* and *y* are positive numbers.

9. $x^{1/2}$ equals: **a.** $\frac{x}{2}$ **b.** $\frac{2}{x}$ **c.** $\sqrt{x}$

10. $x^{2/3}$ equals: **a.** $(\sqrt[3]{x})^2$ **b.** $\sqrt{x^3}$ **c.** $\frac{2\sqrt{x}}{3}$

11. $2x^{-1/2}$ equals: **a.** $\frac{1}{\sqrt{2x}}$ **b.** $\frac{1}{2\sqrt{x}}$ **c.** $\frac{2}{\sqrt{x}}$

12. $(2x)^{-1/2}$ equals: **a.** $\frac{1}{\sqrt{2x}}$ **b.** $\frac{1}{2\sqrt{x}}$ **c.** $\frac{\sqrt{2}}{x}$

13. $-8x^{-1/3}$ equals: **a.** $\frac{-2}{\sqrt[3]{x}}$ **b.** $\frac{-8}{\sqrt[3]{x}}$ **c.** $\frac{-1}{2\sqrt[3]{x}}$

14. $\sqrt[4]{x^3y^{-4}}$ equals: **a.** $\frac{x^{3/4}}{y}$ **b.** $\frac{x^{3/4}}{y^{-1}}$ **c.** $\frac{3x}{4y}$

15. $\sqrt[3]{\frac{x^2}{8y^{-1}}}$ equals: **a.** $\frac{x^{2/3}y}{8}$ **b.** $\frac{x^{2/3}y^{1/3}}{2}$ **c.** $\frac{x^{2/3}}{2y}$

16. $8 \cdot 2^{1/2}$ equals: **a.** $2^{7/2}$ **b.** $16^{1/2}$ **c.** $16\sqrt{2}$

Tell whether each equation is true or false.

17. a. $9^{1/2} + 4^{1/2} = (9 + 4)^{1/2}$
b. $9^{1/2} \cdot 4^{1/2} = (9 \cdot 4)^{1/2}$

18. a. $2^{1/3} + 4^{1/3} = (2 + 4)^{1/3}$
b. $2^{1/3} \cdot 4^{1/3} = (2 \cdot 4)^{1/3}$

19. a. $\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = a + b$
b. $\left(\frac{1}{a} \cdot \frac{1}{b}\right)^{-1} = ab$

20. a. $(\sqrt{a} + \sqrt{b})^2 = a + b$
b. $(\sqrt{a} \cdot \sqrt{b})^2 = ab$

21. a. $(a^{-1} + b^{-1})^{-2} = a^2 + b^2$
b. $(a^{-1} \cdot b^{-1})^{-2} = a^2b^2$

22. a. $(x^{1/3} + y^{1/3})^6 = x^2 + y^2$
b. $(x^{1/3} \cdot y^{1/3})^6 = x^2y^2$

Give the power to which you would raise both sides of each equation in order to solve the equation.

23. $x^{1/2} = 9$

24. $x^{2/3} = 4$

25. $x^{-1/3} = 2$

26. $x^{-3/4} = 8^{-1}$

Tell what steps you would use to solve each equation.

27. $3x^{1/4} = 6$

28. $5x^{-3/2} = 40$

29. $(x - 3)^{-2} = \frac{1}{4}$

30. $(5x)^{-1/2} = 3$

Written Exercises

Simplify.

A

1. $81^{1/2}$
2. $27^{1/3}$
3. $49^{-1/2}$
4. $32^{-1/5}$
5. $4^{3/2}$
6. $27^{2/3}$
7. $16^{3/4}$
8. $25^{-3/2}$
9. $(-125)^{-1/3}$
10. $(-32)^{-3/5}$
11. $4^{-0.5}$
12. $\left(\frac{4}{9}\right)^{-1.5}$
13. $-8^{2/3}$
14. $-9^{3/2}$
15. $(5^{1/3})^{-3}$
16. $(7^{-2/3})^3$
17. $(16^{-5})^{1/20}$
18. $(27^4)^{-1/12}$
19. $(9^{1/2} + 16^{1/2})^2$
20. $(8^{2/3} - 8^{1/3})^3$

Write in exponential form.

21. $\sqrt{x^3y^5}$
22. $\sqrt[3]{p^4q}$
23. $\sqrt{a^{-2}b^3}$
24. $\sqrt[3]{x^6y^{-4}}$
25. $(\sqrt{a^{-2}b})^5$
26. $\sqrt[3]{8b^6c^{-4}}$
27. $\sqrt[4]{\frac{16^3 \cdot a^{-2}}{b^6}}$
28. $\frac{1}{\sqrt[4]{p^4q^{-8}}}$

Express in simplest radical form.

29. $\sqrt[3]{4} \cdot \sqrt[3]{4}$
30. $\sqrt{8} \cdot \sqrt[6]{8}$
31. $\frac{\sqrt[3]{4}}{\sqrt[6]{2}}$
32. $\frac{\sqrt[5]{27^3}}{\sqrt[5]{9^2}}$
33. $\sqrt[10]{32} \div \sqrt[8]{4}$
34. $\sqrt[6]{8^3} \div \sqrt[6]{4^2}$
35. $\sqrt[4]{27} \cdot \sqrt[8]{9}$
36. $\sqrt[4]{128} \cdot \sqrt[8]{256}$

Simplify each expression. Give answers in exponential form.

B

37. $\sqrt{x} \cdot \sqrt[3]{x} \cdot \sqrt[6]{x}$
38. $\sqrt[3]{a^2} \cdot \sqrt[3]{a^4}$
39. $\sqrt[4]{x} \cdot \sqrt[6]{x} \div \sqrt[3]{x}$
40. $((b^{1/2})^{-2/3})^{3/4}$
41. $a^{1/2}(a^{3/2} - 2a^{1/2})$
42. $(x^{3/2} - 2x^{5/2}) \div x^{1/2}$

Solve each equation.

43. a. $a^{3/4} = 8$
    b. $(3x + 1)^{3/4} = 8$
44. a. $y^{-1/2} = 6$
    b. $(3y)^{-1/2} = 6$
45. a. $2y^{-1/2} = 10$
    b. $(2y)^{-1/2} = 10$
46. a. $(9t)^{-2/3} = 4$
    b. $9t^{-2/3} = 4$
47. $(8 - y)^{1/3} = 4$
48. $(3n - 1)^{-2/3} = \frac{1}{4}$
49. $(x^2 + 4)^{2/3} = 25$
50. $(x^2 + 9)^{1/2} = 5$

Mixed Review Exercises

Express in simplest form without negative exponents.

1. $\frac{x}{x-1} - \frac{2}{x^2-1}$
2. $\sqrt{\frac{8x}{y^3}}$
3. $(a^2 + 5) - (4 + a - a^2)$
4. $(-2c^2)(-2c)^2$
5. $\left(\frac{a}{b^2}\right)^{-2} \cdot \left(\frac{b}{a^3}\right)^{-1}$
6. $(2n + 3)(n - 4)$

10-2 Real Number Exponents

Objective To extend the meaning of exponents to include irrational numbers and to define exponential functions.

You have learned the meaning of the power b^x when x is a rational number. Now let's consider the meaning of b^x when x is an irrational number, as in the expression $2^{\sqrt{3}}$.

The graph of $y = 2^x$ for the rational values of x in the table below is shown in Figure 1. In the table irrational values of y are rounded to the nearest tenth.

x	-3	-2	-1	-0.5	0	0.5	1	1.5	2
$y = 2^x$	0.125	0.25	0.5	0.7	1	1.4	2	2.8	4

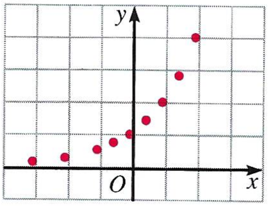


Figure 1

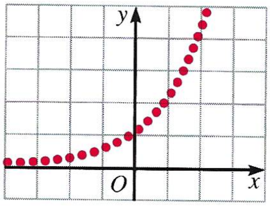


Figure 2

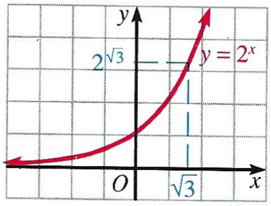


Figure 3

Figure 2 illustrates what happens when you plot many more points of $y = 2^x$ for rational values of x. If the complete graph is to be the smooth curve shown in Figure 3, there must be a point on the curve for every real value of x. In order for there to be no "breaks" in the curve, there must be a point where $x = \sqrt{3}$ and $y = 2^{\sqrt{3}}$.

To find a decimal approximation of $2^{\sqrt{3}}$, look at a sequence of rational numbers x that more and more closely approximate $\sqrt{3}$. Then consider the corresponding sequence of powers 2^x.

sequence of exponents x:	1,	1.7,	1.73,	1.732
sequence of powers 2^x:	2^1,	$2^{1.7}$,	$2^{1.73}$,	$2^{1.732}$
	↓	↓	↓	↓
approximations:	2,	3.2490,	3.3173,	3.3219

As the value of the exponent x gets closer and closer to $\sqrt{3}$, the power 2^x gets closer and closer to a definite positive real number, which is defined to be $2^{\sqrt{3}}$. A scientific calculator can approximate $2^{\sqrt{3}}$ to eight or more significant digits: $2^{\sqrt{3}} \approx 3.3219971$.

The method just discussed can be used to define b^x where $b > 0$ and x is *any real number*. It can be shown in more advanced courses that the laws of exponents also hold for these powers.

Example 1 Simplify: **a.** $\dfrac{6^{\sqrt{2}}}{6^{-\sqrt{2}}}$ **b.** $4^{\pi} \cdot 2^{3-2\pi}$ **c.** $(3^{2\sqrt{2}})^{\sqrt{2}}$

Solution **a.** $\dfrac{6^{\sqrt{2}}}{6^{-\sqrt{2}}} = 6^{\sqrt{2}-(-\sqrt{2})}$

$= 6^{2\sqrt{2}}$, or $36^{\sqrt{2}}$ ***Answer***

b. $4^{\pi} \cdot 2^{3-2\pi} = (2^2)^{\pi} \cdot 2^{3-2\pi}$

$= 2^{2\pi} \cdot 2^{3-2\pi}$

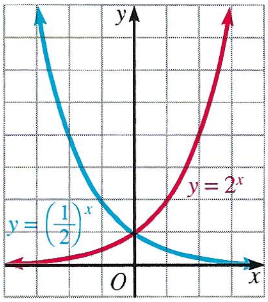
$= 2^{2\pi+(3-2\pi)}$

$= 2^3 = 8$ ***Answer***

c. $(3^{2\sqrt{2}})^{\sqrt{2}} = 3^{2\sqrt{2}\cdot\sqrt{2}} = 3^4 = 81$ ***Answer***

Now that the meaning of exponents has been extended to include all real numbers, *exponential functions* can be defined.

If $b > 0$ and $b \neq 1$, the function defined by $y = b^x$ is called the **exponential function** with base b.



A computer or a graphing calculator may be helpful in exploring the graphs of $y = b^x$ for various values of b.

If $b > 1$, the graph has the general shape shown in red in the diagram. It rises continuously as x increases.

If $0 < b < 1$, the graph has the general shape shown in blue. It falls continuously as x increases.

The graph of every exponential function has 1 for its y-intercept (since $b^0 = 1$) and the x-axis as an asymptote.

For any exponential function,

$$b^p = b^q \text{ if and only if } p = q.$$

This condition expresses the fact that every exponential function is *one-to-one*. A function f is called **one-to-one** if for every p and q in the domain of f, $f(p) = f(q)$ if and only if $p = q$. This is equivalent to saying that any horizontal line intersects the graph of a one-to-one function in at most one point.

In the diagram you can see that any horizontal line intersects the graph of $y = 2^x$ in at most one point. This is also true of the graph of $y = (\frac{1}{2})^x$.

The fact that every exponential function is one-to-one allows you to solve certain kinds of *exponential equations*. An **exponential equation** is an equation in which a variable appears in an exponent.

Solving Exponential Equations

Step 1 Express each side of the equation as a power of the same base.

Step 2 Set the exponents equal and then solve.

Step 3 Check the answer.

Example 2 Solve: **a.** $8^x = \frac{1}{4}$ **b.** $5^{4-t} = 25^{t-1}$

Solution

a. *Step 1* $(2^3)^x = 2^{-2}$

$2^{3x} = 2^{-2}$

Step 2 $3x = -2$

$x = -\frac{2}{3}$

Step 3 *Check:* $8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{4}$ ✓

$\therefore$ the solution set is $\left\{-\frac{2}{3}\right\}$. ***Answer***

b. *Step 1* $5^{4-t} = (5^2)^{t-1}$

$5^{4-t} = 5^{2t-2}$

Step 2 $4 - t = 2t - 2$

$-3t = -6$

$t = 2$

Step 3 The check is left to you.

$\therefore$ the solution set is $\{2\}$.

Answer

Oral Exercises

Which number of the given pair is greater?

1. $7^{\sqrt{2}}, 7^{1.5}$ **2.** $5^{\pi}, 5^{3.14}$ **3.** $4^{2\pi}, 16^3$ **4.** $6^{-\sqrt{2}}, 6^{-\sqrt{3}}$

Simplify.

5. $3^{1+\pi} \cdot 3^{1-\pi}$ **6.** $8^{\sqrt{2}} \cdot 8^{-\sqrt{2}}$ **7.** $\frac{7^{3\sqrt{2}}}{7^{2\sqrt{2}}}$ **8.** $\frac{9^{\sqrt{10}}}{3^{\sqrt{10}}}$

Solve.

9. $3^x = \frac{1}{9}$ **10.** $25^x = 125$ **11.** $4^x = \frac{1}{8}$ **12.** $36^x = \sqrt{6}$

Written Exercises

Simplify.

A **1. a.** $3^{\sqrt{2}} \cdot 3^{\sqrt{2}}$ **b.** $(3^{\sqrt{2}})^2$ **c.** $(3^{\sqrt{2}})^{\sqrt{2}}$ **d.** $\frac{3^{\sqrt{2}+2}}{3^{\sqrt{2}-2}}$

2. a. $7^{\sqrt{3}} \cdot 7^{\sqrt{2}}$ **b.** $(7^{\sqrt{3}})^2$ **c.** $(7^{\sqrt{3}})^{\sqrt{2}}$ **d.** $\frac{7^{\sqrt{3}+2}}{49}$

Simplify.

3. $(10^{\pi})^2$ **4.** $(5^{-\pi})^{-1}$ **5.** $\sqrt{6^{2\pi}}$ **6.** $\sqrt[3]{4^{6\pi}}$

7. $\frac{10^{\sqrt{3}-2}}{10^{\sqrt{3}+2}}$ **8.** $\frac{6^{\sqrt{2}} \cdot 6^{\sqrt{8}}}{6^{3\sqrt{2}}}$ **9.** $[(\sqrt{2})^{\pi}]^0$ **10.** $(\sqrt{3})^{\sqrt{2}}(\sqrt{3})^{-\sqrt{2}}$

Simplify.

11. $(2^{\sqrt{2}})^{-1/\sqrt{2}}$

12. $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$

13. $8^{1.2} \cdot 2^{-3.6}$

14. $\frac{25^{2.4}}{5^{5.8}}$

15. $\frac{(1+\sqrt{3})^{\pi-1}}{(1+\sqrt{3})^{\pi+1}}$

16. $\frac{(\sqrt{2}-1)^{2+\pi}}{(\sqrt{2}-1)^{\pi}}$

17. $\sqrt[4]{\frac{9^{1-\pi}}{9^{1+\pi}}}$

18. $\sqrt{\frac{2^{\sqrt{3}+3}}{8}}$

Solve. If an equation has no solution, say so.

19. $3^x = \frac{1}{27}$

20. $5^x = \sqrt{125}$

21. $8^{2+x} = 2$

22. $4^{1-x} = 8$

23. $27^{2x-1} = 3$

24. $49^{x-2} = 7\sqrt{7}$

25. $4^{2x+5} = 16^{x+1}$

26. $3^{-(x+5)} = 9^{4x}$

27. $25^{2x} = 5^{x+6}$

28. $6^{x+1} = 36^{x-1}$

29. $10^{x-1} = 100^{4-x}$

B 30. $3^{2x} - 6 \cdot 3^x + 9 = 0$

31. $4^{2x} - 63 \cdot 4^x - 64 = 0$

32. $3^{2x} - 10 \cdot 3^x + 9 = 0$

Graph each pair of equations in the same coordinate system. You may wish to verify your graphs on a computer or a graphing calculator.

33. $y = 3^x$ and $y = 3^{x-2}$

34. $y = 3^x - 2$ and $y = 3^x + 2$

35. $y = 2^x$ and $y - 3 = 2^{x-4}$

36. $y = 2^{-x}$ and $y + 3 = 2^{-(x+1)}$

Solve each equation.

C 37. $2^{(2/3)x+1} - 3 \cdot 2^{(1/3)x} - 20 = 0$

38. $2^{2x-1} - 3 \cdot 2^{x-1} + 1 = 0$

39. $2^x - 3^x = 0$

Extra / Growth of Functions

A function f is *increasing* for positive x if $f(x_2) > f(x_1)$ whenever $x_2 > x_1 > 0$. To compare the rate at which two increasing functions grow with x, you look at the ratio $\frac{f(x)}{g(x)}$. The function f will *grow faster* than g if you can make $\frac{f(x)}{g(x)}$ larger than any positive number by taking x large enough.

An important fact about exponential functions is that the exponential function b^x with $b > 1$ grows faster than any polynomial. For example, 2^x grows faster than the polynomial function x^{100}. When $x = 10$ the ratio $\frac{2^x}{x^{100}}$ is very small (approximately 10^{-97}!), but as x increases, the numerator catches up. For $x = 996$ the ratio is about 1, for $x = 1000$ it is about 10, and by taking x still larger we can make the ratio as large as we want. Even an exponential function with a base very close to 1, such as 1.0001^x, will eventually overtake any power x^n, no matter how large n is.

10-3 Composition and Inverses of Functions

Objective To find the composite of two given functions and to find the inverse of a given function.

Consider the squaring function $f(x) = x^2$ and the doubling function $g(x) = 2x$. As the diagram below shows, these two functions can be combined to produce a new function whose value at x is $f(g(x))$, read "f of g of x."

g is the doubling function		f is the squaring function
$3 \longrightarrow$	$g(3) = 6 \longrightarrow$	$f(g(3)) = f(6) = 36$
$5 \longrightarrow$	$g(5) = 10 \longrightarrow$	$f(g(5)) = f(10) = 100$
$x \longrightarrow$	$g(x) = 2x \longrightarrow$	$f(g(x)) = f(2x) = (2x)^2$

Notice that $f(g(x))$ is evaluated by working from the innermost parentheses to the outside. You begin with x, then g doubles x, and then f squares the result.

$$f(g(x)) = f(2x) = (2x)^2$$

The function whose value at x is $f(g(x))$ is called the **composite** of the functions f and g. The operation that combines f and g to produce their composite is called **composition.**

Example 1 If $f(x) = 3x - 5$ and $g(x) = \sqrt{x}$, find the following.

a. $f(g(4))$ **b.** $g(f(4))$
c. $f(g(x))$ **d.** $g(f(x))$

Solution

a. Since $g(4) = \sqrt{4} = 2$, $f(g(4)) = 3 \cdot 2 - 5 = 1$.
b. Since $f(4) = 3 \cdot 4 - 5 = 7$, $g(f(4)) = \sqrt{7}$.
c. $f(g(x)) = f(\sqrt{x}) = 3\sqrt{x} - 5$
d. $g(f(x)) = g(3x - 5) = \sqrt{3x - 5}$

Notice in Example 1 that $f(g(x)) \neq g(f(x))$.

The function $I(x) = x$ is called the **identity function.** It behaves like the multiplicative identity 1.

$$a \cdot 1 = a \quad \text{for all numbers } a$$
$$f(I(x)) = f(x) \quad \text{for all functions } f$$

For the two functions f and g defined in the example at the top of the next page, the composites $f(g(x))$ and $g(f(x))$ are both equal to the identity function.

Example 2 If $f(x) = \frac{x+4}{2}$ and $g(x) = 2x - 4$, find the following.

a. $g(1)$ and $f(g(1))$ **b.** $f(-3)$ and $g(f(-3))$ **c.** $f(g(x))$ **d.** $g(f(x))$

Solution **a.** $g(1) = -2$ and $f(g(1)) = f(-2) = 1$. ***Answer***

Notice that g: $1 \rightarrow -2$ and f: $-2 \rightarrow 1$.

b. $f(-3) = \frac{1}{2}$ and $g(f(-3)) = g\left(\frac{1}{2}\right) = -3$. ***Answer***

Notice that f: $-3 \rightarrow \frac{1}{2}$ and g: $\frac{1}{2} \rightarrow -3$.

Parts (a) and (b) suggest that the functions f and g "undo each other." Parts (c) and (d) prove that this is so for any number x.

c. $f(g(x)) = f(2x - 4) = \frac{2x - 4 + 4}{2} = x$ ***Answer***

d. $g(f(x)) = g\left(\frac{x+4}{2}\right) = 2\left(\frac{x+4}{2}\right) - 4 = x$ ***Answer***

In multiplication, two numbers whose product is the identity 1, such as 2 and 2^{-1}, are called inverses. Similarly, two functions whose composite is the identity I, such as f and g in Example 2, are called *inverse functions*.

Inverse Functions

The functions f and g are **inverse functions** if

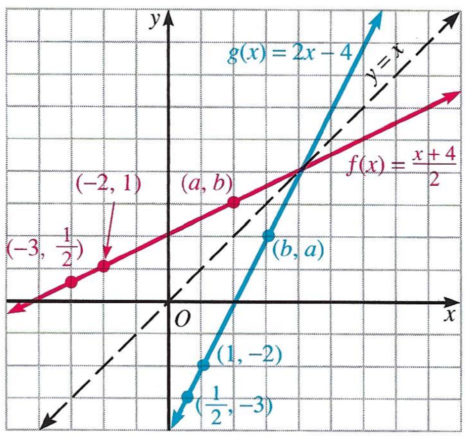
$f(g(x)) = x$ for all x in the domain of g

and $g(f(x)) = x$ for all x in the domain of f.

The inverse of a function f is usually denoted f^{-1}, read "f inverse."

Caution: The superscript -1 in f^{-1} is *not* an exponent. The symbol $f^{-1}(x)$ denotes the value of f inverse at x; it does *not* mean $\frac{1}{f(x)}$.

Suppose that two functions f and g are inverses and that $f(a) = b$, so that (a, b) is on the graph of f. Then $g(b) = g(f(a)) = a$, and the point (b, a) must be on the graph of g. This means every point (a, b) on the graph of f corresponds to a point (b, a) on the graph of g. Therefore, the graphs are *mirror images* of each other with respect to the line $y = x$. You can verify this by drawing graphs of inverse functions on the same axes. A computer or a graphing calculator may be helpful. The diagram shows the inverse functions of Example 2.



Some functions do not have inverse functions. If the reflection of the graph of a function f is itself to be the graph of a function, the graph of f must *not* contain two different points with the same y-coordinate. Therefore, a function has an inverse function if and only if it is one-to-one.

As you learned in Lesson 10-2, every horizontal line intersects the graph of a one-to-one function in at most one point. Therefore, you can use the *horizontal-line test* to tell whether a given function has an inverse function.

Horizontal-Line Test

A function has an inverse function if and only if every horizontal line intersects the graph of the function in *at most* one point.

If a function has an inverse function, you can find it by writing y for $f(x)$, interchanging x and y, and solving for y. Example 3 illustrates.

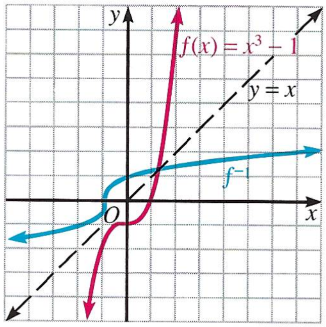
Example 3 Let $f(x) = x^3 - 1$.

a. Graph f and determine whether f has an inverse function. If so, graph f^{-1} by reflecting f across the line $y = x$.

b. Find $f^{-1}(x)$.

Solution

a. The graph of f is shown in red at the right. The graph of f passes the horizontal-line test, so f has an inverse. The graph of f^{-1}, shown in blue, is the reflection of the graph of f across the line $y = x$.



b. Replace $f(x)$ by y: $\quad y = x^3 - 1$

Interchange x and y: $\quad x = y^3 - 1$

Solve for y: $\quad y^3 = x + 1$

$$y = \sqrt[3]{x + 1}$$

$$f^{-1}(x) = \sqrt[3]{x + 1}$$ ***Answer***

Oral Exercises

Suppose $f(x) = 3x$, $g(x) = x + 1$, and $h(x) = x^2 + 2$. Find the following.

1. a. $f(g(3))$ **b.** $f(g(0))$ **c.** $f(g(-6))$ **d.** $f(g(x))$

2. a. $g(f(4))$ **b.** $g(f(5))$ **c.** $g(f(-6))$ **d.** $g(f(x))$

3. a. $f(h(2))$ **b.** $h(f(2))$ **c.** $f(h(x))$ **d.** $h(f(x))$

4. a. $g(h(3))$ **b.** $h(g(3))$ **c.** $g(h(x))$ **d.** $h(g(x))$

5. Find $f^{-1}(x)$.

6. Find $g^{-1}(x)$.

7. Does h^{-1} exist? Why or why not?

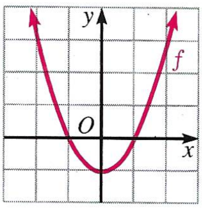
Written Exercises

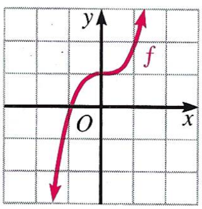
Suppose $f(x) = \frac{x}{2}$, $g(x) = x - 3$, and $h(x) = \sqrt{x}$. Find a real-number value or an expression in x for each of the following. If no real value can be found, say so.

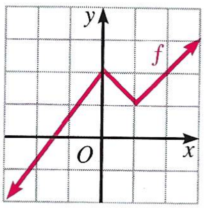
A

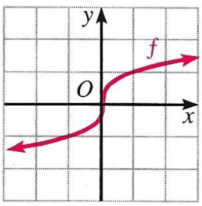
	a.	b.	c.	d.
1.	$f(g(8))$	$f(g(-5))$	$f(g(0))$	$f(g(x))$
2.	$g(f(8))$	$g(f(-5))$	$g(f(0))$	$g(f(x))$
3.	$f(h(9))$	$f(h(4))$	$f(h(-4))$	$f(h(x))$
4.	$h(f(32))$	$h(f(16))$	$h(f(x))$	$f(f(x))$
5.	$h(g(12))$	$h(g(2))$	$h(g(x))$	$h(h(x))$
6.	$g(h(9))$	$g(h(\sqrt{3}))$	$g(h(x))$	$g(g(x))$

Use the horizontal-line test to determine whether each function f has an inverse function. If so, draw a rough sketch of f^{-1} by reflecting f across $y = x$.

7. 

8. 

9. 

10. 

In Exercises 11–14, find $f^{-1}(x)$. Then graph f and f^{-1} in the same coordinate system. You may wish to verify your graphs on a computer or a graphing calculator.

11. $f(x) = 2x - 3$ **12.** $f(x) = \frac{x + 6}{3}$ **13.** $f(x) = x^3$ **14.** $f(x) = \frac{12}{x}$

In Exercises 15–22, graph g and use the horizontal-line test to determine if g has an inverse function. If so, find $g^{-1}(x)$. If g has no inverse, say so. You may wish to verify your graphs on a computer or a graphing calculator.

15. $g(x) = \left(\frac{8}{x}\right)^3$ **16.** $g(x) = \sqrt[3]{2x}$ **17.** $g(x) = x^4$ **18.** $g(x) = |x|$

B **19.** $g(x) = x^2 - x$ **20.** $g(x) = x^3 + 2$ **21.** $g(x) = \sqrt{x^2}$ **22.** $g(x) = (2x + 3)^5$

23. a. Draw the graph of $f(x) = 2^x$ by making a table of values and carefully plotting several points.

b. Draw the graph of f^{-1} on the same coordinate system by reflecting the graph of f in the line $y = x$.

c. Find $f^{-1}(2)$, $f^{-1}(4)$, $f^{-1}(8)$, and $f^{-1}(\frac{1}{2})$.

d. Give the domain and range of f and f^{-1}.

24. The squaring function $f(x) = x^2$ has no inverse function. However, the square root function $g(x) = \sqrt{x}$ *does* have an inverse function. Find this inverse and explain why it is not the same as $f(x) = x^2$. [*Hint:* Compare the domains of f and g^{-1}.]

C 25. Let $f(x) = mx + b$. For what values of m and b will $f(x) = f^{-1}(x)$ for all x?

26. Prove that the graphs of a linear function and its inverse function are never perpendicular.

Mixed Review Exercises

Simplify.

1. $25^{-3/2}$
2. $\sqrt{18} + \sqrt{32}$
3. $(2 - 3i)^2$
4. $(3^{\sqrt{2}})^{2\sqrt{2}}$
5. $\sqrt{-48} \cdot \sqrt{-12}$
6. $\dfrac{2^{\pi-2}}{2^{\pi+2}}$
7. $(-27)^{2/3}$
8. $\dfrac{\sqrt{2} - 3}{1 + \sqrt{2}}$
9. $\dfrac{5^2 - 1}{1 - 3^2}$
10. $16^{1.75}$
11. $\dfrac{4 + 3i}{2 - i}$
12. $4^{1.5} \cdot 2^{-2}$

Self-Test 1

Vocabulary

exponential form (p. 456)
exponential function (p. 460)
one-to-one function (p. 460)
exponential equation (p. 460)
composite (p. 463)
composition (p. 463)
identity function (p. 463)
inverse functions (p. 464)
horizontal-line test (p. 465)

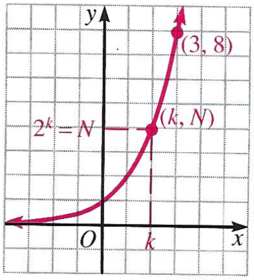
1. Write in exponential form. **Obj. 10-1, p. 455**
   a. $\sqrt[6]{64x^4y^{-2}}$
   b. $\sqrt[3]{6^{1/2}} \cdot \sqrt[6]{6^{-3}}$
2. Write in simplest radical form.
   a. $\left(\dfrac{25}{\sqrt[3]{20}}\right)^{3/2}$
   b. $\sqrt[3]{x^2y^4} \cdot \sqrt{x^3y}$
3. Solve $(x - 2)^{3/5} = 8$.
4. Simplify. **Obj. 10-2, p. 459**
   a. $4^{3/\sqrt{2}} \div 16^{2\sqrt{2}}$
   b. $\dfrac{16^{\sqrt{5}}}{4^{\sqrt{5}}} \cdot 32^{-2\sqrt{5}}$
5. Solve $7^{x-1} = 49^{4-x}$.
6. If $f(x) = 6x + 1$ and $g(x) = \sqrt{x}$, find the following. **Obj. 10-3, p. 463**
   a. $f(g(4))$
   b. $g(f(4))$
   c. $f(g(x))$
   d. $g(f(x))$
7. Show that $f(x) = 3x - 7$ and $g(x) = \dfrac{x + 7}{3}$ are inverse functions.

Logarithmic Functions

10-4 Definition of Logarithms

Objective To define logarithmic functions and to learn how they are related to exponential functions.

The graph of $f(x) = 2^x$ is shown in Figure 1. Since the function passes the horizontal-line test, it has an inverse function. The graph of f^{-1} can be found by reflecting the graph of f across the line $y = x$, as shown in Figure 2.



$f(x) = 2^x$

Figure 1

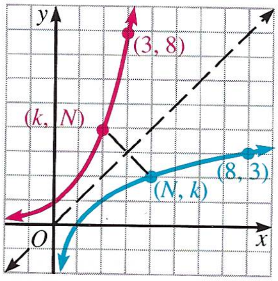
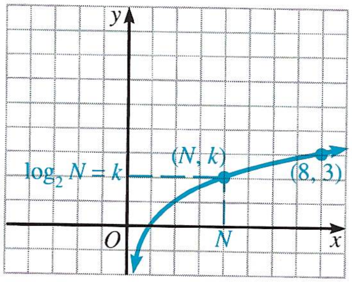


Figure 2



$f^{-1}(x) = \log_2 x$

Figure 3

The inverse of f, the exponential function with base 2, is called the **logarithmic function** with base 2, and is denoted by $f^{-1}(x) = \log_2 x$. (See Figure 3.)

$\log_2 x$ is read "the base 2 logarithm of x" or "log base 2 of x."

Notice that (3, 8) is on the graph of $f(x) = 2^x$ and (8, 3) is on the graph of $f^{-1}(x) = \log_2 x$. Since these functions are inverses, it follows that $2^3 = 8$ means $\log_2 8 = 3$. This and other examples are given below.

Exponential Form		*Logarithmic Form*
$2^3 = 8$	means	$\log_2 8 = 3$
$2^4 = 16$	means	$\log_2 16 = 4$
$2^0 = 1$	means	$\log_2 1 = 0$
$2^{-1} = \frac{1}{2}$	means	$\log_2 \frac{1}{2} = -1$
$2^k = N$	means	$\log_2 N = k$

A base other than 2 can be used with logarithmic functions. In fact the base can be any positive number except 1, since every power of 1 is 1.

Definition of Logarithm

If b and N are positive numbers ($b \neq 1$),

$$\log_b N = k \text{ if and only if } b^k = N.$$

Every positive number has a unique logarithm with base b. Since the base b exponential function is one-to-one, its inverse is one-to-one. Therefore,

$$\log_b M = \log_b N \text{ if and only if } M = N.$$

Several other properties of logarithms are easy to verify. Since the statements $\log_b N = k$ and $b^k = N$ are equivalent, by substitution

$$\log_b b^k = k \qquad \text{and} \qquad b^{\log_b N} = N.$$

Since $b = b^1$ and $1 = b^0$, it follows from the definition of a logarithm that

$$\log_b b = 1 \qquad \text{and} \qquad \log_b 1 = 0.$$

Example 1 Write each equation in exponential form.

a. $\log_6 36 = 2$ **b.** $\log_2 2 = 1$ **c.** $\log_{10} (0.001) = -3$

Solution **a.** $6^2 = 36$ **b.** $2^1 = 2$ **c.** $10^{-3} = 0.001$

Example 2 Write each equation in logarithmic form.

a. $6^0 = 1$ **b.** $8^{-2/3} = \frac{1}{4}$ **c.** $5^{3/2} = 5\sqrt{5}$

Solution **a.** $\log_6 1 = 0$ **b.** $\log_8 \frac{1}{4} = -\frac{2}{3}$ **c.** $\log_5 (5\sqrt{5}) = \frac{3}{2}$

Example 3 Simplify each logarithm.

a. $\log_5 25$ **b.** $\log_2 8\sqrt{2}$ **c.** $2^{\log_2 7}$

Solution **a.** Since $5^2 = 25$, $\log_5 25 = 2$. ***Answer***

b. Let $\log_2 8\sqrt{2} = x$.

Then: $2^x = 8\sqrt{2}$

$2^x = 2^3 \cdot 2^{1/2}$

$2^x = 2^{7/2}$ Set the exponents equal.

$x = \frac{7}{2}$

$\therefore \log_2 8\sqrt{2} = \frac{7}{2}$ ***Answer***

c. Since $b^{\log_b N} = N$, $2^{\log_2 7} = 7$. ***Answer***

Example 4 Solve each equation.

a. $\log_4 x = 3$

b. $\log_x 81 = 4$

Solution

a. Rewrite in exponential form.

$$4^3 = x$$

$$64 = x$$

$\therefore$ the solution set is $\{64\}$. ***Answer***

b. Rewrite in exponential form.

$$x^4 = 81$$

Since $81 = 3^4$,

$$x^4 = 3^4$$

$$x = 3$$

$\therefore$ the solution set is $\{3\}$. ***Answer***

Oral Exercises

Express in exponential form.

1. $\log_2 32 = 5$ **2.** $\log_3 9 = 2$ **3.** $\log_7 \sqrt{7} = \frac{1}{2}$ **4.** $\log_3 \frac{1}{81} = -4$

Express in logarithmic form.

5. $4^3 = 64$ **6.** $9^{3/2} = 27$ **7.** $10^{-2} = 0.01$ **8.** $16^{-3/4} = \frac{1}{8}$

Simplify.

9. $\log_6 36$ **10.** $\log_2 16$ **11.** $\log_{10} 100$ **12.** $\log_3 \frac{1}{9}$

13. $\log_2 2\sqrt{2}$ **14.** $\log_7 1$ **15.** $4^{\log_4 16}$ **16.** $\log_6 (6^5)$

17. $\log_2 10$ lies between the consecutive integers __?__ and __?__.

18. $\log_{10} 101$ lies between the consecutive integers __?__ and __?__.

19. $\log_3 \frac{1}{50}$ lies between the consecutive integers __?__ and __?__.

20. If you solve $\log_2 x^2 = \log_2 100$, then the solution set is {?, ?}.

Written Exercises

Simplify each logarithm.

A **1.** $\log_5 125$ **2.** $\log_4 16$ **3.** $\log_3 81$

4. $\log_6 6$ **5.** $\log_3 1$ **6.** $\log_8 4$

7. $\log_5 \frac{1}{25}$ **8.** $\log_2 \frac{1}{8}$ **9.** $\log_6 6\sqrt{6}$

10. $\log_5 25\sqrt{5}$ **11.** $\log_4 \sqrt{2}$ **12.** $\log_{27} \sqrt{3}$

13. $\log_7 \sqrt[3]{49}$

14. $\log_3 \sqrt[5]{9}$

15. $\log_{1/2} 8$

16. $\log_{1/3} 27$

17. $\log_2 \sqrt[3]{\frac{1}{4}}$

18. $\log_{10} \frac{1}{\sqrt{1000}}$

Solve for x.

19. $\log_7 x = 2$

20. $\log_6 x = 3$

21. $\log_9 x = -\frac{1}{2}$

22. $\log_6 x = 2.5$

23. $\log_4 x = -\frac{3}{2}$

24. $\log_{1/9} x = -\frac{1}{2}$

B 25. $\log_x 27 = \frac{3}{2}$

26. $\log_x 64 = 6$

27. $\log_x 7 = -\frac{1}{2}$

28. $\log_x 7 = 1$

29. $\log_x 1 = 0$

30. $\log_x 2 = 0$

31. a. Show that $\log_2 8 + \log_2 4 = \log_2 32$ by simplifying the three logarithms.
 b. Show that $\log_9 3 + \log_9 27 = \log_9 81$ by simplifying the three logarithms.
 c. State a generalization based on parts (a) and (b).

32. a. Simplify $\log_2 8$ and $\log_8 2$.
 b. Simplify $\log_3 \sqrt{3}$ and $\log_{\sqrt{3}} 3$.
 c. State a generalization based on parts (a) and (b).

33. a. If $f(x) = 6^x$, then $f^{-1}(x) = \underline{\ ?\ }$.
 b. Find $f^{-1}(36)$ and $f^{-1}\left(\frac{1}{\sqrt{6}}\right)$.
 c. Give the domain and range of f and f^{-1}.

34. a. If $g(x) = \log_4 x$, then $g^{-1}(x) = \underline{\ ?\ }$.
 b. Find $g^{-1}(2)$ and $g^{-1}\left(-\frac{3}{2}\right)$.
 c. Give the domain and range of g and g^{-1}.

Sketch the graph of each function and its inverse function in the same coordinate system. Label at least three points on each graph. You may wish to check your graphs on a computer or a graphing calculator.

35. $f(x) = 6^x$

36. $g(x) = \log_4 x$

37. $h(x) = \log_{10} x$

38. $k(x) = \left(\frac{1}{2}\right)^x$

Solve.

C 39. $\log_5 (\log_3 x) = 0$

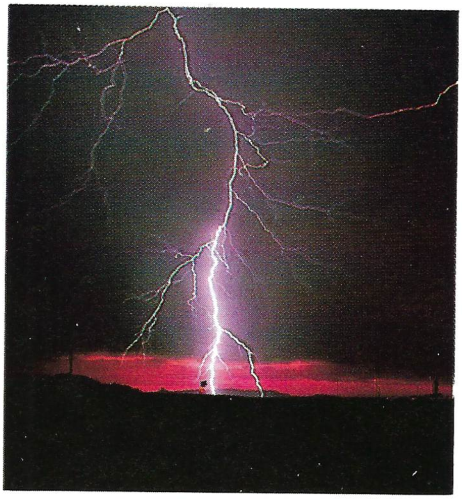
40. $\log_4 (\log_3 (\log_2 x)) = 0$

41. If $0 < b < 1$ and $0 < u < 1$, is $\log_b u$ positive or negative?

42. If $1 < a < b$, is $\log_b (\log_b a)$ positive or negative?

43. The decibel (dB) is used to measure the loudness of sound. Sound that is just barely audible has a decibel level of 0 and an intensity level of 10^{-12} W/m^2 (watts per square meter). Sound painful to the ear has a decibel level of 130 and an intensity level of 10 W/m^2. The formula $D = 10 \log_{10}\left(\frac{I}{I_0}\right)$ gives the decibel level of a sound whose intensity is I, where I_0 is the intensity of barely audible sound. Some decibel levels and intensities are shown in the table below. The intensity is given in watts per square meter.

	Intensity	Decibels
Barely audible sound	10^{-12}	0
Quiet conversation	10^{-9}	30
Heavy traffic	10^{-3}	90
Jet plane (at 20 m)	10^{2}	140



a. The intensity of thunder is 10^{12} times the intensity of barely audible sound. What is the decibel level of thunder?

b. The decibel levels of a subway train entering a station and normal conversation are 100 dB and 60 dB, respectively. How many times as intense as normal conversation is the noise of the subway train?

Historical Note / *Logarithms*

The late sixteenth century was a time of great mathematical activity, particularly in astronomy and navigation. Because of the involved computations that these fields required, there was a need for a method to simplify arithmetic calculation and make multiplication and division easier. The Scottish mathematician John Napier found a way to turn multiplication into addition and division into subtraction, and in 1614 he published a work demonstrating the use of logarithms.

In the nineteenth century, the great French mathematician Laplace said of Napier that by lessening the work of computation his invention doubled the life of the astronomer. Scientists and mathematicians continued to use logarithms until calculators and computers provided more efficient tools.

Napier's logarithms are related to what we today call *natural logarithms*. A year after Napier published his work, the English mathematician Henry Briggs became interested in Napier's invention. It was Briggs who suggested that 10 be the logarithmic base and who introduced the terms *characteristic* and *mantissa*. He compiled an extensive table of logarithms giving 14-place approximations for logarithms of numbers from 1 to 20,000 and 90,000 to 100,000.

10-5 Laws of Logarithms

Objective To learn and apply the basic properties of logarithms.

The laws of exponents can be used to derive the *laws of logarithms*.

Laws of Logarithms

Let b be the base of a logarithmic function ($b > 0$, $b \neq 1$). Let M and N be positive numbers.

1. $\log_b MN = \log_b M + \log_b N$
2. $\log_b \frac{M}{N} = \log_b M - \log_b N$
3. $\log_b M^k = k \log_b M$

Laws 1 and 3 are proved below. The proof of Law 2 is left as Exercise 43.

Proof: Let $\log_b M = x$ and $\log_b N = y$. Therefore, $b^x = M$ and $b^y = N$.

Law 1	*Law 3*
$MN = b^x b^y = b^{x+y}$	$M^k = (b^x)^k = b^{kx}$
$\log_b MN = \log_b b^{x+y}$	$\log_b M^k = \log_b b^{kx}$
$\log_b MN = x + y$	$\log_b M^k = kx$
$\therefore \log_b MN = \log_b M + \log_b N$	$\therefore \log_b M^k = k \log_b M$

Example 1 Express $\log_6 M^2N^3$ in terms of $\log_6 M$ and $\log_6 N$.

Solution

$$\log_6 M^2N^3 = \log_6 M^2 + \log_6 N^3 \qquad \text{Use Law 1.}$$
$$= 2 \log_6 M + 3 \log_6 N \qquad \text{Use Law 3.}$$

Example 2 Express $\log_2 \sqrt{\frac{M}{N^5}}$ in terms of $\log_2 M$ and $\log_2 N$.

Solution

$$\log_2 \sqrt{\frac{M}{N^5}} = \log_2 \left(\frac{M}{N^5}\right)^{1/2}$$
$$= \frac{1}{2} \log_2 \left(\frac{M}{N^5}\right) \qquad \text{Use Law 3.}$$
$$= \frac{1}{2}(\log_2 M - \log_2 N^5) \qquad \text{Use Law 2.}$$
$$= \frac{1}{2}(\log_2 M - 5 \log_2 N) \qquad \text{Use Law 3.}$$

The process illustrated in Examples 1 and 2 can also be reversed.

Example 3 Express as a single logarithm.

a. $\log_{10} p + 3 \log_{10} q$ **b.** $4 \log_{10} p - 2 \log_{10} q$

Solution

a. $\log_{10} p + 3 \log_{10} q = \log_{10} p + \log_{10} q^3$ Use Law 3.

$= \log_{10} pq^3$ Use Law 1.

b. $4 \log_{10} p - 2 \log_{10} q = \log_{10} p^4 - \log_{10} q^2$ Use Law 3.

$= \log_{10} \frac{p^4}{q^2}$ Use Law 2.

You can often use the laws of logarithms to compute the numerical value of a logarithm from given logarithms.

Example 4 If $\log_{10} 2 = 0.30$ and $\log_{10} 3 = 0.48$, find the following.

a. $\log_{10} 18$ **b.** $\log_{10} \frac{20}{3}$ **c.** $\log_{10} 5$ **d.** $\log_{10} \left(\frac{1}{\sqrt[3]{2}}\right)$

Solution

a. Since $18 = 2 \cdot 3^2$,

$$\begin{aligned}\log_{10} 18 &= \log_{10} 2 + 2 \log_{10} 3\\ &= 0.30 + 2(0.48)\\ &= 1.26\end{aligned}$$ ***Answer***

b. Since $\frac{20}{3} = \frac{2 \cdot 10}{3}$,

$$\begin{aligned}\log_{10} \frac{20}{3} &= \log_{10} 2 + \log_{10} 10 - \log_{10} 3\\ &= 0.30 + 1 - 0.48\\ &= 0.82\end{aligned}$$ ***Answer***

c. Since $5 = \frac{10}{2}$,

$$\begin{aligned}\log_{10} 5 &= \log_{10} 10 - \log_{10} 2\\ &= 1 - 0.30\\ &= 0.70\end{aligned}$$ ***Answer***

d. Since $\frac{1}{\sqrt[3]{2}} = 2^{-1/3}$,

$$\begin{aligned}\log_{10} \left(\frac{1}{\sqrt[3]{2}}\right) &= -\frac{1}{3} \log_{10} 2\\ &= -\frac{1}{3}(0.30)\\ &= -0.10\end{aligned}$$ ***Answer***

You can use the laws and properties of logarithms to solve logarithmic equations. *Because the logarithm of a variable expression is defined only if the expression is positive, be sure to check the answers you obtain.*

Example 5 Solve $\log_3 x + \log_3 (x - 6) = 3$ for x.

Solution

$$\log_3 x + \log_3 (x - 6) = 3$$
$$\log_3 x(x - 6) = 3 \qquad \text{Use Law 1.}$$
$$\log_3 (x^2 - 6x) = 3$$

Change to exponential form: $x^2 - 6x = 3^3$

$$x^2 - 6x - 27 = 0$$
$$(x + 3)(x - 9) = 0$$
$$x = -3 \quad \text{or} \quad x = 9$$

Check: If $x = -3$, $\log_3 x$ is not defined. So -3 is *not* a root.

If $x = 9$, $\log_3 x + \log_3 (x - 6) = \log_3 9 + \log_3 3$
$$= \log_3 (9 \cdot 3)$$
$$= \log_3 27$$
$$= 3 \checkmark$$

$\therefore$ the solution set is $\{9\}$. ***Answer***

Oral Exercises

Express each logarithm in terms of $\log_3 M$ and $\log_3 N$.

1. $\log_3 M^4$ **2.** $\log_3 N^6$ **3.** $\log_3 M^4N$ **4.** $\log_3 \left(\frac{M}{N^3}\right)$

5. $\log_3 \left(\frac{1}{M}\right)$ **6.** $\log_3 \sqrt{M}$ **7.** $\log_3 \sqrt[3]{N^2}$ **8.** $\log_3 \left(\frac{1}{N\sqrt{N}}\right)$

Express as a logarithm of a single number or expression.

9. $\log_a 3 + \log_a 4$

10. $\log_a 7 - \log_a 5$

11. $4 \log_a 2$

12. $2 \log_a 9$

13. $\frac{1}{2} \log_a 36$

14. $-\log_a \frac{1}{6}$

15. $\log_b 3 + \log_b 5 + \log_b 2$

16. $\log_b 6 + \log_b 5 - \log_b 2$

17. $2 \log_b p + \log_b q$

18. $\log_b x - 3 \log_b y$

19. $\frac{1}{2} \log_b r + \frac{1}{2} \log_b s$

20. $\frac{1}{2}(\log_b x - \log_b y)$

Let $c = \log_3 10$ and $d = \log_3 5$. Express the following in terms of c and d.

21. $\log_3 50$ **22.** $\log_3 500$ **23.** $\log_3 250$ **24.** $\log_3 2$

Written Exercises

Express each logarithm in terms of $\log_2 M$ and $\log_2 N$.

A

1. $\log_2 M^6N^3$
2. $\log_2 (MN)^4$
3. $\log_2 M\sqrt{N}$
4. $\log_2 \sqrt[3]{M^2N}$
5. $\log_2 \frac{M^4}{N^3}$
6. $\log_2 \left(\frac{M}{N}\right)^7$
7. $\log_2 \sqrt{\frac{M}{N^3}}$
8. $\log_2 \frac{1}{MN}$

If $\log_{10} 9 = 0.95$ and $\log_{10} 2 = 0.30$ (accurate to two decimal places), find the following.

9. $\log_{10} 81$
10. $\log_{10} \frac{9}{2}$
11. $\log_{10} \sqrt{2}$
12. $\log_{10} 3$
13. $\log_{10} 8$
14. $\log_{10} 36$
15. $\log_{10} \frac{20}{9}$
16. $\log_{10} 900$
17. $\log_{10} \frac{1}{9}$
18. $\log_{10} \frac{1}{2000}$
19. $\log_{10} \sqrt[3]{\frac{2}{9}}$
20. $\log_{10} 162$

Express as a logarithm of a single number or expression.

21. $5 \log_4 p + \log_4 q$
22. $\log_{10} x - 4 \log_{10} y$
23. $4 \log_3 A - \frac{1}{2} \log_3 B$
24. $\log_5 M + \frac{1}{4} \log_5 N$

B

25. $\log_2 M + \log_2 N + 3$
26. $\log_5 x - \log_5 y + 2$
27. $1 - 3 \log_5 x$
28. $\frac{1 + \log_9 x}{2}$

Simplify.

29. $2 \log_{10} 5 + \log_{10} 4$
30. $2 \log_3 6 - \log_3 4$
31. $\log_4 40 - \log_4 5$
32. $\log_4 3 - \log_4 48$

Solve each equation.

33. $\log_a x = 2 \log_a 3 + \log_a 5$
34. $\log_a x = \frac{3}{2} \log_a 9 + \log_a 2$
35. $\log_b (x + 3) = \log_b 8 - \log_b 2$
36. $\log_b (x^2 + 7) = \frac{2}{3} \log_b 64$
37. $\log_a x - \log_a (x - 5) = \log_a 6$
38. $\log_a (3x + 5) - \log_a (x - 5) = \log_a 8$
39. $\log_2 (x^2 - 9) = 4$
40. $\log_3 (x + 2) + \log_3 6 = 3$
41. If $f(x) = \log_2 x$ and $g(x) = 4^x$, find: **a.** $f(g(3))$ **b.** $g\left(f\left(\frac{1}{2}\right)\right)$ **c.** $f(g^{-1}(16))$
42. If $f(x) = 3^x$ and $g(x) = \log_9 x$, find: **a.** $f(f(-1))$ **b.** $g\left(g\left(\frac{1}{81}\right)\right)$ **c.** $f^{-1}(g(9))$
43. Prove the second law of logarithms.
44. Simplify $4^{\log_2 (2^{\log_2 5})}$.

Solve each equation.

C **45.** $\log_5 (\log_3 x) = 0$

46. $\log_2 (\log_4 x) = 1$

47. $\log_6 (x + 1) + \log_6 x = 1$

48. $\log_{10} (x + 6) + \log_{10} (x - 6) = 2$

49. $\frac{1}{2} \log_a (x + 2) + \frac{1}{2} \log_a (x - 2) = \frac{2}{3} \log_a 27$

50. $2 \log_3 x - \log_3 (x - 2) = 2$

51. $\log_b (x - 1) + \log_b (x + 2) = \log_b (8 - 2x)$

Mixed Review Exercises

Solve.

1. $\log_2 x = -\frac{1}{2}$

2. $x^3 - 7x + 6 = 0$

3. $2^{x+3} = 4^{x-1}$

4. $\sqrt{x} + 2 = x$

5. $3^x = \frac{\sqrt{3}}{9}$

6. $3(2x - 1) = 5x + 4$

7. $\frac{x}{x + 1} - 1 = \frac{1}{x}$

8. $\log_x 8 = \frac{3}{2}$

9. $(x - 2)^2 = 5$

If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x + 1}$, find each of the following.

10. $f(-2)$

11. $g(3)$

12. $f(g(1))$

13. $g(f(-1))$

Self-Test 2

Vocabulary logarithmic function (p. 468)
logarithm (p. 469)
laws of logarithms (p. 473)

1. Write in exponential form. **Obj. 10-4 p. 468**

a. $\log_3 81 = 4$

b. $\log_6 216 = 3$

2. Write in logarithmic form.

a. $5^4 = 625$

b. $25^{3/2} = 125$

3. Evaluate each expression.

a. $\log_2 4^{3/2}$

b. $4^{\log_4 12}$

4. Solve $\log_b 27 = 3$.

5. Write $\log_2 (M^5N^6)^{1/3}$ in terms of $\log_2 M$ and $\log_2 N$. **Obj. 10-5 p. 473**

6. If $\log_{10} 5 = 0.70$, find $\log_{10} 0.04$.

7. Solve $\log_a x + \log_a (x - 2) = \log_a 3$.

Check your answers with those at the back of the book.

10-6 Applications of Logarithms

Objective To use common logarithms to solve equations involving powers and to evaluate logarithms with any given base.

Logarithms were invented to simplify difficult calculations. Because of the decimal nature of our number system, it is easiest to work with base 10 logarithms. These are called **common logarithms.** When common logarithms are used in calculations, the base 10 is usually not written. For example, log 6 means $\log_{10} 6$.

Most scientific calculators have a key marked "log" that gives the common logarithm of a number. You also can find the common logarithm of a number by using Table 3 on page 812.

Although calculators and computers have replaced logarithms for doing heavy computational work, logarithms still provide the best, or even the only, method of solution for many types of problems. Some of these will be shown later in this lesson and in the next.

If you have a calculator, you should practice finding the logarithms of a few numbers and then skip ahead to the paragraph preceding Example 2.

Without a calculator, you will need to use the table of logarithms (Table 3), a portion of which is shown below.

N	0	1	2	3	4	5	6	7	8	9
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962

The table gives the logarithms of numbers between 1 and 10 rounded to four decimal places. The decimal point is omitted. To find an approximation for $\log x$ for $1 < x < 10$, find the first two digits of x in the column headed N and the third digit in the row to the right of N. For example, to find log 2.36, look for 23 under N and move across row 23 to the column headed 6, where you find 3729. Therefore, $\log 2.36 \approx 0.3729$.

To find an approximation for the logarithm of a positive number greater than 10 or less than 1, write the number in scientific notation (see page 221) and use Law 1: $\log MN = \log M + \log N$. For example,

$$\begin{aligned}\log 236 &= \log (2.36 \times 10^2) \\ &= \log 2.36 + \log 10^2 && \text{Use Law 1.} \\ &= \log 2.36 + 2 \\ &\approx 0.3729 + 2, \text{ or } 2.3729 && \text{Use Table 3.}\end{aligned}$$

The example shows that the common logarithm of a number can be written as the sum of an integer, called the **characteristic,** and a nonnegative number less than 1, called the **mantissa,** which can be found in a table of logarithms.

Example 1 Find each logarithm. Use Table 3.

a. log 3.8 **b.** log 97,500 **c.** log 0.000542

Solution **a.** Find 38 under N; then read across to the column headed 0.

$\therefore$ log 3.8 = 0.5798 ***Answer***

b. Write 97,500 in scientific notation and use laws of logarithms.

$$\log 97{,}500 = \log (9.75 \times 10^4) = \log 9.75 + \log 10^4$$

The mantissa for 9.75 is 0.9890. The characteristic is 4.

$\therefore$ log 97,500 = 0.9890 + 4 = 4.9890 ***Answer***

c. Write 0.000542 in scientific notation and use laws of logarithms.

$$\log 0.000542 = \log (5.42 \times 10^{-4}) = \log 5.42 + \log 10^{-4}$$

The mantissa for 5.42 is 0.7340. The characteristic is -4.

$\therefore$ $\log 0.000542 = 0.7340 + (-4) = -3.2660$ ***Answer***

The logarithms in the tables, and therefore the answers, are approximations. However, it is a common practice to use =, as in Example 1, rather than ≈.

If $\log y = a$, then the number y is sometimes called the **antilogarithm** of a. The value of y can be found by using a calculator or log tables.

Example 2 Find y to three significant digits if:

a. $\log y = 0.8995$ **b.** $\log y = 2.4825$

Solution 1 Using a Calculator On some calculators you find antilogarithms by using the inverse function key with the logarithmic function key. On many others, you can use the 10^x key as shown.

a. If $\log_{10} y = 0.8995$, then $y = 10^{0.8995} = 7.93$. ***Answer***

b. $y = 10^{2.4825} = 304$ ***Answer***

Solution 2 Using Tables

a. In the body of Table 3 find the value closest to 0.8995, that is, 0.8993. This is the entry in the row labeled 79 and in the column labeled 3.

$\therefore y = 7.93$ ***Answer***

b. First find the number that has 0.4825 as its mantissa. The table entry with mantissa closest to 0.4825 is 3.04.

$$\begin{aligned} \log y &= 2.4825 = 0.4825 + 2 \\ &= \log 3.04 + \log 10^2 \\ &= \log (3.04 \times 10^2) \\ \log y &= \log 304 \end{aligned}$$

$\therefore y = 304$ ***Answer***

In the previous example the answers were rounded to three significant digits. For most problems the answers you find using Table 3 will be accurate to three significant digits.

Example 3 Find the value of each expression to three significant digits.

a. $\sqrt[5]{493}$ **b.** $(0.173)^6$

Solution 1 Using a Calculator

a. Since $\sqrt[5]{493} = 493^{1/5} = 493^{0.2}$, use the power key, y^x.

$y^x = 493^{0.2} = 3.46$ ***Answer***

b. $y^x = (0.173)^6 = 0.0000268$ ***Answer***

Solution 2 Using Tables

a. Let $x = \sqrt[5]{493}$.

$\log x = \frac{1}{5} \log 493$

$\log x = \frac{1}{5}(2.6928)$

$\log x = 0.5386$

$\therefore \quad x = 3.46$ ***Answer***

b. $(0.173)^6 = (1.73 \times 10^{-1})^6$
$= (1.73)^6 \times 10^{-6}$

Let $x = (1.73)^6$. Then:

$\log x = 6 \log 1.73$

$\log x = 1.428$

$x = 26.8$

$\therefore (0.173)^6 = 26.8 \times 10^{-6}$
$= 0.0000268$ ***Answer***

Whether or not you use a calculator, you need the properties of logarithms to solve the exponential equation given in Example 4. The solution to Example 4 is given in *calculation-ready form,* where the next step is to obtain a decimal approximation using a calculator or a table.

Example 4 Solve $3^{2x} = 5$.

a. Give the solution in calculation-ready form.

b. Give the solution as a decimal with three significant digits.

Solution

a.

$3^{2x} = 5$ Take logarithms of both sides.

$\log 3^{2x} = \log 5$

$2x \log 3 = \log 5$ Use laws of logarithms to simplify.

$x = \frac{\log 5}{2 \log 3}$ Solve for x.

$\therefore$ the solution in calculation-ready form is $\frac{\log 5}{2 \log 3}$. ***Answer***

b. Find log 5 and log 3 with a calculator or table.

$$x = \frac{0.6990}{2(0.4771)} = 0.733$$

$\therefore$ to three significant digits, the solution is 0.733. ***Answer***

If you know the base b logarithm of a number and wish to find its base a logarithm, you can use the following formula:

Change-of-Base Formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

The proof of this formula is left as Exercise 42 on page 482.

Example 5 Find $\log_4 7$.

Solution Use the change-of-base formula letting $b = 10$:

$$\log_4 7 = \frac{\log_{10} 7}{\log_{10} 4} = \frac{0.8451}{0.6021} = 1.404 \quad \textit{Answer}$$

Oral Exercises

Use a calculator or Table 3 to find each logarithm.

1. $\log 4.05$
2. $\log 40.5$
3. $\log 405$
4. $\log 0.405$
5. $\log 8.36$
6. $\log 83.6$
7. $\log 0.836$
8. $\log 0.0836$

Use a calculator or Table 3 to find x to three significant digits.

9. $\log x = 0.6072$
10. $\log x = 0.9212$
11. $\log x = 1.9212$
12. $\log x = 0.9212 - 1$

Solve for x in calculation-ready form.

13. $2^x = 7$
14. $9^x = 8$
15. $8^{2x} = 3$
16. $3^{-x} = 7$

Written Exercises

Use a calculator or Table 3 to find the value of each expression to three significant digits.

A

1. $(1.06)^{10}$
2. $(10.6)^{10}$
3. $(0.38)^5$
4. $(347)^{1.5}$
5. $(12.7)^{5/2}$
6. $\sqrt[6]{786}$
7. $\sqrt[5]{(81.2)^4}$
8. $\sqrt[3]{(412)^2}$

Use a calculator or Table 3 to find x to three significant digits.

9. $\log x = 0.8531$
10. $\log x = 0.4065$
11. $\log x = 2.84$
12. $\log x = 1.605$
13. $\log x = -1.8$
14. $\log x = -2.91$

Solve each equation.

a. **Give the solution in calculation-ready form.**
b. **Give the solution to three significant digits.**

15. $3^x = 30$ **16.** $5^t = 10$ **17.** $5.6^x = 56$ **18.** $(1.02)^x = 2$

19. $30^{-x} = 5$ **20.** $12^{2x} = 1000$ **21.** $3.5^{2t} = 60$ **22.** $\frac{4^{2-t}}{3} = 7$

Solve each equation *without* using a calculator or logarithms. (See Example 2 of Lesson 10-2, page 461.)

23. $4^x = 8\sqrt{2}$ **24.** $3^x = \sqrt[5]{9}$ **25.** $125^x = 25\sqrt{5}$ **26.** $8^x = 16\sqrt[3]{2}$

Unlike Exercises 15–26, Exercises 27–34 are *not* exponential equations. Solve each equation using a calculator or logarithms. Give answers to three significant digits.

Sample Solve $x^{2/3} = 12$.

Solution 1. Raise each side to the $\frac{3}{2}$ power. You find that $x = 12^{3/2}$.
2. To simplify $12^{3/2}$.
(a) Use a calculator and the y^x key to obtain $x = 41.6$, or
(b) Use logarithms to obtain $\log x = \frac{3}{2} \log 12 = 1.619$. Then find the antilogarithm: $x = 41.6$. ***Answer***

27. $x^{2/5} = 34$ **28.** $x^{2/3} = 50$ **29.** $\sqrt[3]{x^4} = 60$ **30.** $\sqrt[5]{x^3} = 900$

B **31.** $2x^5 = 100$ **32.** $\frac{\sqrt[5]{x}}{9} = 7$ **33.** $(3y - 1)^6 = 80$ **34.** $\sqrt[3]{4t + 3} = 8.15$

Find each logarithm. Use the change-of-base formula.

35. $\log_2 9$ **36.** $\log_6 8$ **37.** $\log_3 40$ **38.** $\log_7 \frac{1}{2}$

Solve for x to three significant digits.

39. $3^{2x} - 7 \cdot 3^x + 10 = 0$ **40.** $3^{2x} - 7 \cdot 3^x + 12 = 0$

41. **a.** Simplify $\log_7 49$ and $\log_{49} 7$.
b. Simplify $\log_2 8$ and $\log_8 2$.
c. How are $\log_b a$ and $\log_a b$ related? Give a convincing argument to justify your answer.

C **42.** Derive the change-of-base formula. (*Hint:* Let $\log_a x = y$, so that $x = a^y$.)

10-7 Problem Solving: Exponential Growth and Decay

Objective To use exponential and logarithmic functions to solve growth and decay problems.

Suppose an investment of P dollars earns 8% interest compounded annually. Then the value of the investment will be multiplied by 1.08 each year because:

$$\begin{aligned}\text{Value at end of year} &= 100\%(\text{value at beginning of year}) + \text{interest earned}\\ &= 1.00(\text{value at beginning of year}) + 0.08(\text{value at beginning of year})\\ &= 1.08(\text{value at beginning of year})\end{aligned}$$

From this equation you can obtain the following table of values.

Time in years	0	1	2	3	...	t
Value in dollars	P	$P(1.08)$	$P(1.08)^2$	$P(1.08)^3$	...	$P(1.08)^t$
	×1.08	×1.08	×1.08			

Because the value of the investment in t years is given by the exponential function $A = P(1.08)^t$, we say that the investment has **exponential growth** and that the annual *growth rate* is 8%.

If the 8% annual interest is compounded quarterly (that is, compounded four times per year), the quarterly growth rate is 2%. Then

$$\text{value at end of } q \text{ quarters} = P(1.02)^q$$

and

$$\text{value at end of } t \text{ years} = P(1.02)^{4t}.$$

This result can be generalized as follows.

Compound Interest Formula

If an amount P (called the *principal*) is invested at an annual interest rate r compounded n times a year, then in t years the investment will grow to an amount A given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where r is expressed as a decimal.

Example 1 How long will it take an investment of $1000 to triple in value if it is invested at an annual rate of 12% compounded quarterly?

(Solution is on the next page.)

Solution Use the compound interest formula. Let $P = 1000$, $A = 3000$, $r = 0.12$, and $n = 4$.

$$3000 = 1000\left(1 + \frac{0.12}{4}\right)^{4t}$$
$$3 = (1.03)^{4t}$$
$$\log 3 = \log (1.03)^{4t}$$
$$\log 3 = 4t \log 1.03$$
$$t = \frac{\log 3}{4 \log 1.03} = \frac{0.4771}{4(0.0128)}$$
$$t = 9.3 \text{ (years)}$$

Since the interest is compounded *quarterly,* you cannot give 9.3 as an answer. Since 9.3 is closer to $9\frac{1}{4}$ than $9\frac{1}{2}$, the investment will triple in approximately $9\frac{1}{4}$ years. ***Answer***

Investing money is just one example of exponential growth. Another example is the growth of a population. If a population now has N_0 people and is growing at a rate of 1.4% per year, then in t years the population will have $N_0(1.014)^t$ people.

A growth rate of 1.4% per year approximately doubles the population every 50 years, since $N_0(1.014)^{50} \approx 2N_0$. From this equation you can see that $1.014 \approx 2^{1/50}$. Therefore, the number N of people in the population t years from now can be written in either of two ways:

$N = N_0(1.014)^t$ (The population grows 1.4% per year.)

$N = N_0 \cdot 2^{t/50}$ (The population doubles every 50 years.)

Doubling-Time Growth Formula

If a population of size N_0 doubles every d years (or hours, or days, or any other unit of time), then the number N in the population at time t is given by

$$\mathbf{N = N_0 \cdot 2^{t/d}.}$$

Example 2 A certain bacteria population doubles in size every 12 hours. By how much will it grow in 2 days?

Solution Use the doubling-time growth formula. Let $t = 48$ hours (2 days) and let the doubling time $d = 12$ hours.

$$N = N_0 \cdot 2^{t/d}$$
$$N = N_0 \cdot 2^{48/12} = N_0 \cdot 2^4 = 16N_0$$

$\therefore$ the population grows by a factor of 16 in 2 days. ***Answer***

While investments and populations grow exponentially, the value of a car decreases exponentially, or experiences **exponential decay.** If a car loses 25% of its value each year, then its value N at the end of a year is

$$N = 0.75 \times (\text{value at beginning of year}).$$

Therefore, after t years a car initially valued at N_0 dollars will have a value of

$$N = N_0(0.75)^t.$$

One frequently studied example of exponential decay is the decay of a radioactive substance. Over a period of time, some of the substance will change to a different element. The *half-life* of the radioactive substance is the amount of time it takes until exactly half of the original substance remains unchanged. For example, the half-life of radioactive radium is 1600 years. Therefore, every 1600 years half of the radium decays.

Time in years	0	1600	3200	4800	. . .	t
Amount left	N_0	$N_0 \cdot \frac{1}{2}$	$N_0 \cdot \left(\frac{1}{2}\right)^2$	$N_0 \cdot \left(\frac{1}{2}\right)^3$	. . .	$N_0 \cdot \left(\frac{1}{2}\right)^{t/1600}$

The half-life decay formula is very similar to the doubling-time growth formula.

Half-Life Decay Formula

If an amount N_0 has a half-life h, then the amount remaining at time t is

$$N = N_0\left(\frac{1}{2}\right)^{t/h}.$$

Example 3 The half-life of carbon-14 (C-14) is 5730 years. How much of a 10.0 mg sample will remain after 4500 years?

Solution Use the half-life decay formula. Let $N_0 = 10.0$, $h = 5730$, and $t = 4500$.

$$N = N_0\left(\frac{1}{2}\right)^{t/h}$$

$$N = 10.0\left(\frac{1}{2}\right)^{4500/5730}$$

Using a calculator: $N = 5.80$ (to three significant digits)

If a calculator is not available, take logarithms of both sides:

$$\log N = \log 10.0 + \frac{4500}{5730} \log 0.5$$

$$\log N = 1 + (0.7853)(-0.3010)$$

$$\log N = 0.7636$$

$$N = 5.80$$

$\therefore$ 5.80 mg of the original C-14 remains after 4500 years. ***Answer***

Oral Exercises

Complete each table.

1.

Item	Cost now	Annual growth rate	Cost in t years
Movie ticket	\$5	10%	?
Sweater	\$40	5%	?
Sneakers	?	?	$60(1.07)^t$

2.

Item	Cost now	Annual decay rate	Value in t years
Car	\$10,000	30%	?
Bike	\$300	20%	?
Skis	?	?	$250(0.95)^t$

3.

Population size now	Doubling time	Size in t years
1000	9 years	?
3 million	25 years	?
?	?	$7500 \cdot 2^{t/12}$

4.

Amount of radioactive substance now	Half-life in days	Amount of substance in t days
80	8	?
1200	30	?
?	?	$500\left(\frac{1}{2}\right)^{t/3}$

Problems

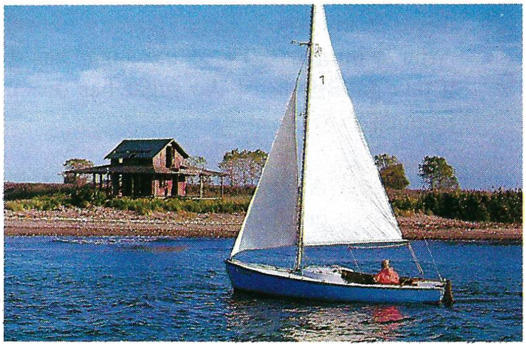
Solve. Give money answers in dollars and cents and all other answers to three significant digits. A calculator may be helpful.

A

1. One thousand dollars is invested at 12% interest compounded annually. Determine how much the investment is worth after:
   a. 1 year **b.** 2 years **c.** 3 years **d.** 10 years
2. One hundred dollars is invested at 7.2% interest compounded annually. Determine how much the investment is worth after:
   a. 1 year **b.** 5 years **c.** 10 years **d.** 20 years
   e. Use your answers to parts (a)–(d) to estimate the doubling time for the investment.
3. Redo Exercise 1 assuming that the interest is compounded quarterly.
4. Redo Exercise 1 assuming that the interest is compounded monthly.

5. The value of a new \$12,500 automobile decreases 20% per year. Find its value after:

   a. 1 year **b.** 2 years
   c. 3 years **d.** 10 years



6. The value of a new \$3500 sailboat decreases 10% per year. Find its value after:

   a. 1 year **b.** 5 years
   c. 10 years **d.** 20 years

7. A certain population of bacteria doubles every 3 weeks. The number of bacteria in the population is now N_0. Find its size in:

   a. 6 weeks **b.** 15 weeks **c.** W weeks

8. A culture of yeast doubles in size every 20 min. The size of the culture is now N_0. Find its size in:

   a. 1 hour **b.** 12 hours **c.** 1 day

9. The half-life of carbon-14 is approximately 6000 years. Determine how much of 100 kg of this substance will remain after:

   a. 12,000 years **b.** 24,000 years **c.** y years

10. The radioactive gas radon has a half-life of approximately $3\frac{1}{2}$ days. About how much of a 100 g sample will remain after 1 week?

B 11. How long will it take you to double your money if you invest it at a rate of 8% compounded annually?

12. How long will it take you to triple your money if you invest it at a rate of 6% compounded annually?

13. Savings institutions sometimes use the term "effective yield" to describe the interest rate compounded *annually* (once a year) that is equivalent to a given rate. The effective yield can be found by computing the value of \$1 at the end of the year at the given rate. For example, at 8% compounded quarterly, the value A of \$1 at the end of a year is

$$A = 1\left(1 + \frac{0.08}{4}\right)^4 = (1.02)^4.$$

Evaluating A directly (without using logs), you find that $A = 1.0824$ to four decimal places. Therefore, the original \$1 has grown in value by \$0.0824, an increase of 8.24%. The effective yield is 8.24% per year. Find the effective yield of 12% compounded quarterly.

14. Bank A offers 6% interest compounded monthly. Bank B offers 6.1% compounded quarterly. Which bank pays more interest per year?

15. One of the many by-products of uranium is radioactive plutonium, which has a half-life of approximately 25,000 years. If some of this plutonium is encased in concrete, what fraction of it will remain after 100 years? after a million years?

16. Sugar is put into a large quantity of water and the mixture is stirred. After 2 minutes 50% of the sugar has dissolved. How much longer will it take until 90% of the sugar has dissolved? (Use the half-life decay formula.)

Sample A baseball card increased in value from \$50 to \$500 in 15 years. Find its average annual rate of appreciation.

Solution Let r be the annual rate of appreciation.

Then
$$50(1 + r)^{15} = 500$$
$$(1 + r)^{15} = 10$$
$$1 + r = 10^{1/15} \approx 1.17$$
$$\therefore r \approx 0.17 = 17\%$$

17. A gold coin appreciated in value from \$100 to \$238 in 8 years. Find the average annual rate of appreciation.

18. Ten years ago Michael paid \$250 for a rare 1823 stamp. Its current value is \$1000. Find the average annual rate of growth.

19. A new car that cost \$12,000 decreased in value to \$4000 in 5 years. Find the average annual rate of depreciation.

20. A tractor that 4 years ago cost \$8000, now is worth only \$3200. Find the average annual rate of depreciation.

C **21.** One million dollars is invested at 6.4% interest. Find the value of the investment after one year if the interest is compounded:

a. quarterly **b.** daily **c.** hourly

d. Why do you think your answers to parts (a)–(c) are so close in value?

22. A colony of bacteria has 6.5×10^6 members at 8 A.M. and 9.75×10^6 members at 10:30 A.M. Find its population at noon.

Mixed Review Exercises

If the domain of each function is $D = \{1, 2, 4\}$, find the range.

1. $f(x) = \log_4 x$ **2.** $g(x) = |x - 3|$ **3.** $h(x) = (\sqrt{2})^x$

4. $F(x) = x^2 - 3x + 2$ **5.** $G(x) = \dfrac{x}{x + 2}$ **6.** $H(x) = x^{1/2}$

Simplify.

7. $\log_3 12 - 2 \log_3 2$ **8.** $(4^{1/3})^{3/2}$ **9.** $\dfrac{2^{-1} - 2^{-2}}{1 + 2^{-1}}$

10. $\dfrac{8^{0.5}}{2^{1.5}}$ **11.** $\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2}$ **12.** $\log_6 24 + 2 \log_6 3$

10-8 The Natural Logarithm Function

Objective To define and use the natural logarithm function.

In advanced work in science and mathematics, the most important logarithm function is the **natural logarithm function.** Its base is the irrational number e, which has the approximate value 2.71828. The **natural logarithm** of x is sometimes denoted by $\log_e x$, but more often by $\ln x$.

The number e is defined to be the limiting value of $\left(1 + \frac{1}{n}\right)^n$ as n becomes larger and larger. Using a calculator, you can make a table and show that as n gets very large, the value of this expression approaches a number a bit larger than 2.718.

n	100	1000	10,000	100,000
$\left(1 + \frac{1}{n}\right)^n$	2.70481	2.71692	2.71815	2.71827

The exercises of this lesson are just like exercises in previous lessons, except that the symbol $\ln x$ is used instead of $\log_b x$. Example 1 illustrates.

Example 1

Working with base 2 logs

1. If $\log_2 x = 5$, then $x = 2^5$.
2. If $2^x = 7$, then $x = \log_2 7$.
3. $\log_2 2^5 = 5$ and $2^{\log_2 7} = 7$

Working with base e logs

1. If $\ln x = 5$, then $x = e^5$.
2. If $e^x = 7$, then $x = \ln 7$.
3. $\ln e^5 = 5$ and $e^{\ln 7} = 7$

Example 2

a. Simplify $\ln \frac{1}{e^2}$.

b. Write as a single logarithm: $2 \ln 5 + \ln 4 - 3$.

Solution

a. $\ln \frac{1}{e^2} = \ln e^{-2} = -2 \ln e = -2 \cdot 1 = -2$ ***Answer***

b. $2 \ln 5 + \ln 4 - 3 = \ln 5^2 + \ln 4 - \ln e^3$

$= \ln \frac{5^2 \cdot 4}{e^3} = \ln \frac{100}{e^3}$ ***Answer***

Example 3 Solve: **a.** $\ln x = 2$ **b.** $\ln \frac{1}{x} = 2$

Solution

a. $x = e^2$ ***Answer***

b. $\frac{1}{x} = e^2$

$x = \frac{1}{e^2}$, or e^{-2} ***Answer***

Example 4 Solve $e^{2x} = 9$.

Solution 1

$$e^{2x} = 9$$

$$2x = \ln 9 \qquad \text{Rewrite in logarithmic form.}$$

$$x = \frac{1}{2} \ln 9$$

$$x = \ln 9^{1/2} = \ln 3 \quad \textit{Answer}$$

Solution 2

$$e^{2x} = 9$$

$$\ln e^{2x} = \ln 9 \qquad \text{Take the natural log of both sides of the equation.}$$

$$2x \cdot \ln e = \ln 9$$

$$2x \cdot 1 = \ln 9$$

$$x = \frac{1}{2} \ln 9 = \ln 9^{1/2} = \ln 3 \quad \textit{Answer}$$

Oral Exercises

Give each equation in exponential form.

1. $\ln 4 = 1.39$
2. $\ln \frac{1}{4} = -1.39$
3. $\ln e = 1$

Give each equation in logarithmic form.

4. $e^2 = 7.39$
5. $e^{-2} = 0.14$
6. $e^{1/5} = 1.22$

Simplify.

7. $\ln \frac{1}{e}$
8. $\ln e^{12}$
9. $\ln \sqrt{e}$

Solve.

10. $\ln x = 5$
11. $\ln x = \frac{1}{3}$
12. $e^x = 3$
13. $x = \ln e^{3/4}$
14. $e^{\ln x} = 10$
15. $\ln x = -\frac{1}{2}$
16. Approximate to three decimal places the value of $\left(1 + \frac{1}{5000}\right)^{5000}$.

Written Exercises

Write each equation in exponential form.

A

1. $\ln 8 = 2.08$
2. $\ln 100 = 4.61$
3. $\ln \frac{1}{3} = -1.10$
4. $\ln \frac{1}{e^2} = -2$

Write each equation in logarithmic form.

5. $e^3 = 20.1$
6. $e^7 = 1097$
7. $e^{1/2} = 1.65$
8. $\sqrt[3]{e} = 1.40$

Simplify. If the expression is undefined, say so.

9. $\ln e^2$
10. $\ln e^{10}$
11. $\ln \frac{1}{e^3}$
12. $\ln \frac{1}{\sqrt{e}}$
13. $\ln 1$
14. $\ln 0$
15. $e^{\ln 5}$
16. $e^{\ln 0.5}$

Write as a single logarithm.

17. $\ln 3 + \ln 4$
18. $\ln 8 - \ln 2$
19. $2 \ln 3 - \ln 5$
20. $\ln 7 + \frac{1}{2} \ln 9$
21. $\frac{1}{3} \ln 8 + \ln 5 + 3$
22. $4 \ln 2 - \ln 3 - 1$

Solve for *x*. Leave answers in terms of *e*.

23. $\ln x = 3$
24. $\ln \frac{1}{x} = 2$
25. $\ln (x - 4) = -1$
26. $\ln |x| = 1$
27. $\ln x^2 = 9$
28. $\ln \sqrt{x} = 3$

Solve for *x*. Leave answers in terms of natural logarithms.

29. $e^x = 2$
30. $e^{-x} = 3$
31. $e^{2x} = 25$
32. $e^{3x} = 8$
33. $e^{x-2} = 2$
34. $\frac{1}{e^x} = 7$

Solve. Leave answers in terms of *e* or natural logarithms.

35. $\sqrt{e^x} = 3$
36. $e^{-2x} = 0.2$
37. $(e^x)^5 = 1000$
38. $3e^{2x} + 2 = 50$
39. $\ln (\ln x) = 0$
40. $|\ln x| = 1$
41. $\ln x + \ln (x + 3) = \ln 10$
42. $2 \ln x = \ln (x + 1)$
43. $e^{2x} - 7e^x + 12 = 0$

Give the domain and range of each function.

B 44. $f(x) = \ln x$
45. $f(x) = \ln |x|$
46. $f(x) = \ln x^2$
47. $f(x) = \ln (x - 5)$

48. Graph $y = \ln x$ and $y = e^x$ in the same coordinate system.

49. Graph $y = 2^x$, $y = e^x$, and $y = 3^x$ in the same coordinate system.

C 50. Express in terms of *e* the approximate value of each expression when *n* is very large.

a. $\left(1 + \frac{1}{n}\right)^{5n}$
b. $\left(1 + \frac{2}{n}\right)^{n}$
c. $\left(\frac{n}{n + 1}\right)^{2n}$

51. Refer to the compound interest formula on page 483.

a. Show that if interest is compounded daily, so that *n* is quite large, then $A \approx Pe^{rt}$.

b. Use part (a) to find the amount of money that you would have after 1 year if you invest \$1000 at 6% interest compounded daily.

Calculator Key-In

If your calculator has an e^x or y^x key, or a natural logarithm ($\ln x$) and an inverse function key, you can evaluate powers of e.

1. If you invest P dollars at an annual interest rate r compounded daily, then the value of the investment t years later will be about Pe^{rt}. (Note that the rate is expressed as a decimal. Therefore, if the rate is 6%, then $r = 0.06$.) Find the value of a \$1000 investment at 6% interest compounded daily after:

   a. 1 year **b.** 2 years **c.** 10 years

2. Which investment is worth more?

   Investment A: \$1000 at 8% compounded daily after 10 years

   Investment B: \$2000 at 8% compounded daily after 5 years

3. **a.** Complete the following table for $y = e^{-x^2}$.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y											

   b. Use the table to sketch the graph of $y = e^{-x^2}$. (*Hint:* The graph is symmetric about the y-axis.) This graph is closely related to the so-called bell-shaped curve so important in statistics.

4. **a.** Evaluate $\left(1 + \frac{1}{n}\right)^n$ when n is a million.

   b. Compare your answer with the following approximation for e: 2.718281828.

5. The notation $n!$ (read "n factorial") represents the product of the integers from 1 to n.

$$n! = 1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot n$$

   A very important number in mathematics is approximated by the sum

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}.$$

   Find this sum and tell what important number it approximates.

Challenge

A friend is thinking of an integer between 1 and 1,000,000. You are to guess the number by asking your friend questions that require only yes-or-no answers. What is the minimum number of questions that you must ask to be sure of guessing the number? How should you guess?

Self-Test 3

Vocabulary

common logarithms (p. 478)
characteristic (p. 479)
mantissa (p. 479)
antilogarithm (p. 479)
calculation-ready form (p. 480)
change-of-base formula (p. 481)
exponential growth (p. 483)
compound interest formula (p. 483)
doubling-time formula (p. 484)
exponential decay (p. 485)
half-life (p. 485)
half-life decay formula (p. 485)
natural logarithm function (p. 489)
natural logarithm (p. 489)

Solve. Give answers to three significant digits.

1. $x^{2/3} = 75$
2. $2^{x-2} = 14$
3. $10^{2x} = 5^{x+2}$ **Obj. 10-6, p. 478**
4. You invest \$2000 in a bank offering 10% interest compounded quarterly. Find the value of your investment after a total of five years' growth. **Obj. 10-7, p. 483**
5. If $e^{3x+2} = 5$, find x in terms of natural logarithms. **Obj. 10-8, p. 489**

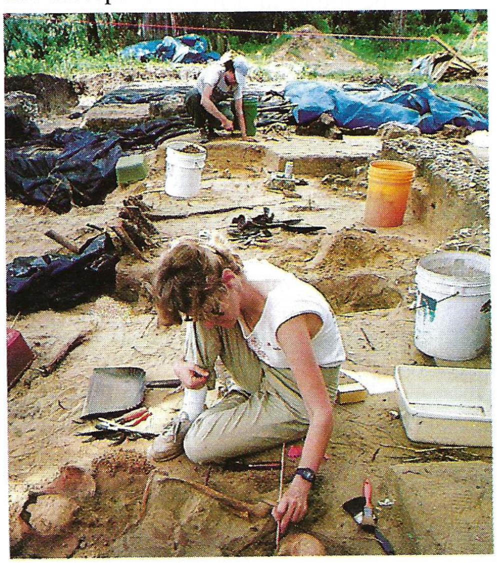
Check your answers with those at the back of the book.

Application / *Radiocarbon Dating*

When archaeologists uncover a piece of bone or wood from the site of a dig, they are often able to date the time when the animal or tree was alive by using radiocarbon dating. Most of the carbon in Earth's atmosphere is the isotope C-12, but a small amount is the radioactive isotope C-14. The C-14 decays with a half-life of 5730 years, but the ratio of C-14 to C-12 in the atmosphere remains approximately constant because the C-14 is restored by cosmic ray bombardment of atoms in the atmosphere. The carbon taken in and used by growing plants and animals contains the same fraction of C-14 as the atmosphere. When an organism dies, the amount of C-12 it contains remains the same, but the C-14 decays. Archaeologists therefore use the half-life decay formula,

$$N = N_0 \cdot \left(\frac{1}{2}\right)^{t/h}, \qquad \text{or } \frac{N}{N_0} = \left(\frac{1}{2}\right)^{t/h},$$

where N is the amount of C-14 in the sample discovered, N_0 is the amount of C-14 in the sample when it was alive, and h is the half-life of C-14, 5730 years.



The quantity N is found by measuring the radioactivity of the sample. N_0 is not known, but it can be estimated by finding the total amount of carbon in the sample and assuming that the atmospheric C-14 to C-12 ratio was the same when the sample was alive as it is now, about 1 to 10^{12}. The age t of the sample can then be computed. For example, if the amount N of C-14 is $\frac{1}{8}$ of the estimated amount N_0 that was in the sample when it was alive, then

$$\frac{N}{N_0} = \frac{1}{8} = \left(\frac{1}{2}\right)^{t/5730}$$

$$2^{-3} = 2^{-t/5730}$$

$$-3 = -\frac{t}{5730}$$

$$t = 3 \cdot 5730 = 17{,}190$$

Therefore, the age of the sample is approximately 17,000 years.

The dates found by radiocarbon dating can often be checked against other dating methods. For example, *dendrochronology* (establishing a chronology by correlating sequences of growth rings in wood) has dated some very old samples of long-lived tree species. A piece of wood from a bristle-cone pine has been dated in this way to 6000 B.C. Radiocarbon dating of this sample gave a date of 5500 B.C., showing that 8000 years ago the atmospheric C-14 to C-12 ratio was greater than it is now. (Can you see why a greater ratio then gives a younger date?) Estimates of such atmospheric changes made by comparing tree-ring and radiocarbon dates can be used to obtain more accurate radiocarbon dates. With such corrections these dates are usually accurate to within two centuries for dates up to 10,000 years ago.

Exercises

1. The amount of C-14 in a sample of wood is 45% of the amount that would be found in a living sample with the same total carbon content. How many years have passed since the sample was part of a living tree? Give your answer to two significant digits.

2. A scientist finds that a bone sample contains C-14 and C-12 in the ratio 0.08 to 10^{12}.

a. What is the ratio of C-14 in the sample now (N) to C-14 in the sample when it was part of a living animal (N_0)? Assume that the atmospheric C-14 fraction was the same then as it is now.

b. How many years have passed since the animal was living? Give your answer to two significant digits.

3. Using tree-ring chronology it is found that 400 years have passed since a given wood sample was part of a living tree.

a. What percent of the original C-14 can a scientist expect to find today?

b. What percent would a scientist 200 years from now expect to find?

Give your answers to the nearest tenth of a percent.

Chapter Summary

1. The meaning of rational exponents is defined as follows:

$$b^{p/q} = (\sqrt[q]{b})^p = \sqrt[q]{b^p}$$

Rational exponents can be used to define powers with real exponents, and the laws of exponents hold for real exponents.

2. The function defined by $y = b^x$ is called the *exponential function* with base b, where $b > 0$ and $b \neq 1$.

3. The function whose value at x is $f(g(x))$ is called the *composite* of f and g. The operation that combines f and g to produce their composite is called *composition*.

4. *Inverse functions* are functions that "undo" one another. The functions f and g are inverse functions if

$$f(g(x)) = x \text{ for all } x \text{ in the domain of } g$$

and $$g(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

5. The function $y = \log_b x$ is called the *logarithmic function* with base b, where $b > 0$ and $b \neq 1$. This function is the inverse of the exponential function with base b.

6. *Logarithms* are defined by the relationship

$$\log_b N = k \text{ if and only if } b^k = N,$$

where b and N are positive numbers and $b \neq 1$.

7. In working with logarithms, you can use the *laws of logarithms*.

Law 1 $\quad \log_b MN = \log_b M + \log_b N$

Law 2 $\quad \log_b \dfrac{M}{N} = \log_b M - \log_b N$

Law 3 $\quad \log_b M^k = k \log_b M$

8. There are certain types of equations for which logarithms provide the best, or even the only, method of solution. Logarithms are used in studying exponential equations and their applications to such occurrences as compound interest, population growth, and radioactive decay. Here are some related formulas:

Compound interest formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ (See page 483.)

Doubling-time growth formula: $N = N_0 \cdot 2^{t/d}$ (See page 484.)

Half-life decay formula: $N = N_0 \cdot \left(\frac{1}{2}\right)^{t/h}$ (See page 485.)

9. An important function used in science and mathematics is the *natural logarithm function* denoted by $\ln x$ or $\log_e x$. Its base is the irrational number e, which is approximately 2.71828.

Chapter Review

Write the letter of the correct answer.

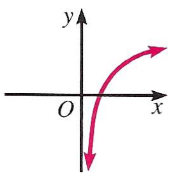
1. For positive x and y, which expression is equivalent to $(x^{1/2} - y^{1/2})^2$? **10-1**
   a. $x - y$ **b.** $x + 2\sqrt{xy} + y$ **c.** $x - 2\sqrt{xy} + y$

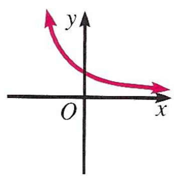
2. Simplify $\frac{2}{\sqrt[6]{8}}$.
   a. $\frac{1}{\sqrt{2}}$ **b.** $\frac{1}{2}$ **c.** $\sqrt{2}$ **d.** $\sqrt[3]{4}$

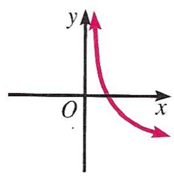
3. Solve $25^{x+2} = 5^{3x-3}$. **10-2**
   a. 2 **b.** 3 **c.** 4 **d.** 7

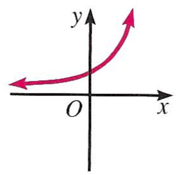
4. Simplify $\left(\frac{2^{1+\sqrt{2}}}{2^{1-\sqrt{2}}}\right)^{\sqrt{2}}$.
   a. 4 **b.** $2^{\sqrt{2}}$ **c.** 8 **d.** 16

5. Which of the following could be the graph of $y = \left(\frac{1}{3}\right)^x$?

a.


b.


c.


d.


6. If $f(x) = 3x - 2$ and $g(x) = 2x - 1$, then $g(f(x))$ is equal to: **10-3**
   a. $6x - 4$ **b.** $6x - 5$ **c.** $6x^2 - x - 2$ **d.** $6x - 3$

7. Are the functions f and g inverse functions if $f(x) = \frac{5}{3}x + 1$ and $g(x) = \frac{3(x-1)}{5}$?
   a. Yes **b.** No

8. If $f(x) = 9x - 7$, find the function g that is the inverse of f.
   a. $g(x) = 7x + 9$ **b.** $g(x) = \frac{1}{9}x + 7$ **c.** $g(x) = \frac{x+7}{9}$

9. Simplify $\log_4 64$. **10-4**
   a. $\frac{1}{3}$ **b.** 3 **c.** 16 **d.** 4

10. Between which two numbers is $\log_5 150$?
    a. 0 and 1 **b.** 1 and 2 **c.** 2 and 3 **d.** 3 and 4

11. Simplify $\log_6\left(\frac{36}{6^{-10}}\right)$. **10-5**
    a. 12 **b.** -8 **c.** -20 **d.** -5

12. If $\log_b z = \frac{1}{3}\log_b x + \log_b y$, write z in terms of x and y.
    a. $y\sqrt[3]{x}$ **b.** $(x + y)^{1/3}$ **c.** $(xy)^{1/3}$ **d.** $\frac{x}{3} + y$

13. Solve $10^{5t} = 2$. **10-6**

a. $\frac{2}{5}$ **b.** $\frac{\log_{10} 2}{\log_{10} 5}$ **c.** $5 \log_{10} 2$ **d.** $\frac{1}{5} \log_{10} 2$

14. If 40 mg of a radioactive substance decays to 5 mg in 12 min, find the half-life, in minutes, of the substance. **10-7**

a. 2 min **b.** 4 min **c.** 6 min **d.** 8 min

15. Simplify $\ln \frac{1}{e^3}$. **10-8**

a. 3 **b.** -3 **c.** 0 **d.** 1

16. Solve $e^{2x-1} = 3$.

a. $\ln 3 + 1$ **b.** $\frac{\ln 3}{2x - 1}$ **c.** $\frac{\ln 3 + 1}{2}$ **d.** $1 - \ln 3$

Chapter Test

1. Find the value of x. **10-1**

a. $\left(\frac{1}{16}\right)^{-3/4} = x$ **b.** $27^x = 81$

2. Simplify.

a. $\sqrt[3]{\sqrt{125y^6}}$ **b.** $(64^{2/3} + 27^{2/3})^{3/2}$

3. Solve $4^{x-2} = 8^{\pi+1} \div 8^{\pi-1}$. **10-2**

4. Suppose $f(x) = 2x - 1$ and $g(x) = x^2 + 4$. Find: **10-3**

a. $f(g(-2))$ **b.** $g(f(x))$

5. Suppose that $f(x) = \sqrt[3]{x - 1}$ and $g(x) = x^3 + 1$. Show that f and g are inverse functions.

6. **a.** Write in logarithmic form using the base 2: $32^{3/5} = 8$. **10-4**

b. Write in exponential form: $\log_{16}\left(\frac{1}{64}\right) = -\frac{3}{2}$.

7. Simplify: **a.** $6^{\log_6 3}$ **b.** $\log_3 27^{\sqrt{2}}$

8. Solve: **a.** $\log_3 x = \log_3 12 + \log_3 2 - \log_3 6$ **10-5**

b. $\log_4 (x - 6) + \log_4 x = 2$

9. If $\log_{10} 2 = 0.301$ and $\log_{10} 3 = 0.477$, find the following.

a. $\log_{10} 8$ **b.** $\log_{10} 12$ **c.** $\log_{10} 15$

10. Express t in terms of common logarithms: $5^{3t} = 2$. **10-6**

11. The population of a certain colony of bacteria doubles every 5 hours. How long will it take for the population to triple? Give the answer to two significant digits. **10-7**

12. Write x in terms of e: $\ln x^2 = 8$. **10-8**

13. Write as a single natural logarithm: $\frac{1}{2} - \ln 7$.

Mixed Problem Solving

Solve each problem that has a solution. If a problem has no solution, explain why.

A

1. How much iodine should be added to 50 mL of a 10% iodine solution to obtain a 25% solution?
2. When the reciprocal of a real number is subtracted from 1, the difference is equal to the number. Find the number.
3. The number of representatives to Congress from California is equal to the sum of the numbers of representatives from New York and North Carolina. Together the three states have 90 representatives. The number from New York is 23 more than the number from North Carolina. How many representatives does each state have?
4. When two meshed gears revolve, their speeds vary inversely as the numbers of teeth they have. A gear with 36 teeth runs at 200 r/min (revolutions/min). Find the speed of a 60-tooth gear meshed with it.
5. The difference of two numbers is 6. Find their minimum product.
6. A rectangular garden has perimeter 44 m. A walkway 2 m wide surrounds the garden. The combined area of the garden and walkway is 224 m^2. Find the dimensions of the garden.
7. When Susan had annual incomes of $15,000 and $18,000, she saved $600 and $900, respectively. If the amount saved is a linear function of the amount earned, how much would she save when her income becomes $24,000?
8. The *eagle* is a 10-dollar gold United States coin minted from 1795 to 1933. Find all the possible ways a person could receive $25 using eagles and quarter-eagles.

B

9. The area of a circle is directly proportional to the square of its circumference. A circle with circumference 14π has area 49π. Find a general formula relating the circumference and the area of a circle.
10. Find three consecutive integers such that the square of the sum of the first two is 80 more than the sum of the squares of the last two.
11. Two water purification plants can process a day's water supply in 5 h. Operating alone, the smaller plant would need 4 h more than the larger one would. How long would each need to do the job alone?
12. The value of a certain car decreases at a rate of 12% annually. About how long does it take for the car to be worth half its original value?
13. If Rachel jogs for awhile at 8 km/h and then walks at 5 km/h, she travels 23 km. If she reverses the amount of time spent jogging and walking, she travels 6 km farther. Find her total traveling time.

Preparing for College Entrance Exams

Strategy for Success

If you skip a question, don't forget to leave the line for that answer blank on your answer sheet. To ensure an accurate pairing between questions and answers, every so often compare the numbering in your test booklet with the one on your answer sheet.

Decide which is the best of the choices given and write the corresponding letter on your answer sheet.

1. Which equation represents the set of all points that are equidistant from $(-7, 4)$ and $(3, 6)$?

(A) $x - 5y = -27$ **(B)** $5x - y = -15$ **(C)** $5x + y = -5$
(D) $5x - y = 15$ **(E)** $x + 5y = 23$

2. The graph of $9x^2 - 18 = y^2$ is

(A) a parabola **(B)** a circle **(C)** an ellipse
(D) a hyperbola **(E)** a point

3. Which expression below is equivalent to $-3 \log_8 4$?

(A) $\frac{\sqrt[5]{32^4}}{32}$ **(B)** $\log_5 \sqrt{5} - \log_5 5\sqrt{5}$ **(C)** $\log 1 - \log 0.01$
(D) $\sqrt{-4}$ **(E)** $\ln \frac{1}{e^2}$

4. An ellipse with equation $\frac{x^2}{16} + \frac{y^2}{36} = 1$ has which point as a focus?

(A) $(0, -2\sqrt{3})$ **(B)** $(0, 2\sqrt{5})$ **(C)** $(-2\sqrt{5}, 0)$
(D) $(2\sqrt{13}, 0)$ **(E)** $(0, 6)$

5. If $f(x) = \frac{4}{x} - 1$, find $f^{-1}(7)$ if possible.

(A) $\frac{1}{2}$ **(B)** $-\frac{3}{7}$ **(C)** 4 **(D)** 0 **(E)** f^{-1} does not exist.

6. Solve $27^{2t-1} = 81^{t+2}$.

(A) 3 **(B)** -3 **(C)** $-\frac{1}{2}$ **(D)** -2 **(E)** $\frac{11}{2}$

7. Find the number of real solutions of this system:

$$x^2 + 16y^2 = 25$$
$$xy - 3 = 0$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

8. Find the z-coordinate of the solution of this system:

$$5x - 3y + z = 5$$
$$4x + 3y - 2z = -4$$
$$2x - 3y - 7z = 13$$

(A) 0 **(B)** 1 **(C)** -1 **(D)** 2 **(E)** -2