NAME ______________________ DATE ______________

3 Linear Equations and Functions

3–1 Open Sentences in Two Variables

Objective: To find solutions of open sentences in two variables and to solve problems involving open sentences in two variables.

Vocabulary

Open sentence in two variables An equation or inequality involving two variables.

Domain of x The set whose members may serve as replacements for x.

Ordered pair A pair of numbers having a definite order. A solution to an open sentence in the two variables x and y is given as the ordered pair (x, y). Example: The ordered pair $(3, 4)$ is a solution to the equation $x + 2y = 11$.

Solution set The set of all solutions (ordered pairs) that satisfy an open sentence.

Example 1 Solve the equation $3x + 2y = 6$ if the domain of x is $\{-1, 0, 2\}$.

Solution Substitute each value in the domain of x in the equation. Then solve for y. You're looking for solutions of the form $(-1, ?)$, $(0, ?)$, and $(2, ?)$.

$x = -1$	$x = 0$	$x = 2$
$3x + 2y = 6$	$3x + 2y = 6$	$3x + 2y = 6$
$3(-1) + 2y = 6$	$3(0) + 2y = 6$	$3(2) + 2y = 6$
$-3 + 2y = 6$	$0 + 2y = 6$	$6 + 2y = 6$
$2y = 9$	$2y = 6$	$2y = 0$
$y = \frac{9}{2}$	$y = 3$	$y = 0$

$\therefore$ the solution set is $\left\{\left(-1, \frac{9}{2}\right), (0, 3), (2, 0)\right\}$.

Solve each equation if the domain of x is (a) $\{-1, 0, 2\}$, and (b) $\{-2, 1, 3\}$.

1. $x - 2y = 1$ **2.** $-x + 2y = 0$ **3.** $3x - 2y = 5$ **4.** $2x - \frac{1}{3}y = 1$

Complete each ordered pair to form a solution of the equation.

5. $x - y = 3$ $(-2, \underline{?})$, $(0, \underline{?})$, $(\underline{?}, -1)$

6. $2x + 5y = 20$ $(0, \underline{?})$, $(5, \underline{?})$, $(\underline{?}, 0)$

7. $\frac{1}{2}x + y = 1$ $(-2, \underline{?})$, $(0, \underline{?})$, $(\underline{?}, 5)$

8. $x + 3y = 2$ $(-1, \underline{?})$, $(2, \underline{?})$, $\left(\underline{?}, \frac{5}{3}\right)$

Example 2 Find the value of k so that $(2, -1)$ satisfies the equation $kx - 2y = k$.

Solution

$$\begin{aligned} kx - 2y &= k \\ k(2) - 2(-1) &= k \\ 2k + 2 &= k \\ 2k &= k - 2 \\ k &= -2 \end{aligned}$$

Find the value of k so that the ordered pair satisfies the equation.

9. $3x + y = k$; $(-1, 2)$ **10.** $5x - ky = 4$; $(2, -3)$ **11.** $3x + ky = k$; $(-1, -2)$

3–1 Open Sentences in Two Variables *(continued)*

Example 3 A child's bank contains \$1.15 in dimes and quarters. Find all possibilities for the number of each type of coin in the bank.

Solution

Step 1 The problem asks for the number of dimes and quarters whose total value is \$1.15.

Step 2 Let d = number of dimes and q = number of quarters.

Step 3 The total value of d dimes is $10d$ cents and of q quarters is $25q$ cents.
Write an equation expressing the total value of the coins in cents:

$$10d + 25q = 115$$

Step 4 Solve the equation for one variable, say d, in terms of the other.

$$\begin{aligned} 10d + 25q &= 115 \\ 2d + 5q &= 23 \\ 2d &= 23 - 5q \\ d &= \frac{23 - 5q}{2} \end{aligned}$$

Remember: The number of each type of coin must be a whole number. If q is even, then $23 - 5q$ is odd and d is not a whole number. Therefore, q must be an odd whole number.

q	$\frac{23 - 5q}{2}$	d	(q, d)
1	$\frac{23 - 5(1)}{2}$	9	(1, 9)
3	$\frac{23 - 5(3)}{2}$	4	(3, 4)

If $q \geq 5$, then $d < 0$.

Step 5 The check is left for you.

∴ the bank can contain 9 dimes and 1 quarter, or 4 dimes and 3 quarters.

In each problem (a) choose two variables to represent the numbers asked for, (b) write an open sentence relating the variables, and (c) solve the open sentence and give the answer. (Include solutions in which one of the variables is equal to zero.)

12. A cashier needs to refund \$90 using \$10 and \$20 bills. Find all the possibilities for the number of each type of bill the cashier could use.

13. Terry has 85 cents in nickels and quarters. Find all the possibilities for the number of each type of coin that Terry could have.

14. Sam needs \$1.75 in change to make a long-distance phone call from a pay phone. He has a roll of quarters and a roll of dimes. Find all the possibilities for the number of each type of coin that Sam could use.

Mixed Review Exercises

Solve each open sentence and graph each solution set that is not empty.

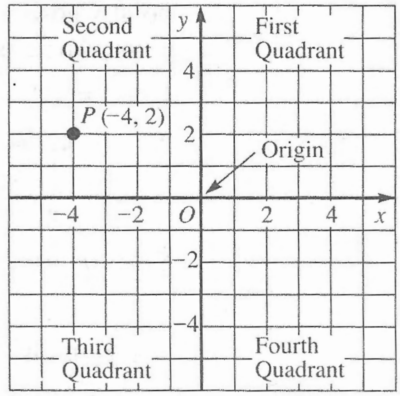
1. $-2 \leq 2a - 1 < 4$ **2.** $|2 - b| < 1$ **3.** $2c + 3 \leq 5c - 6$

NAME ______________________ DATE ______________

3–2 Graphs of Linear Equations in Two Variables

Objective: To graph a linear equation in two variables.

Vocabulary



Plane rectangular coordinate system The intersection of a vertical number line (y-axis) and a horizontal number line (x-axis). A rectangular coordinate system is also called a *Cartesian coordinate system* or an *xy-coordinate plane*.

Quadrant One of the four regions formed by the intersecting axes in a rectangular coordinate system.

Coordinates The ordered pair of real numbers associated with each point in the xy-coordinate plane.
Example: The coordinates of point P are $(-4, 2)$.

x-coordinate (or abscissa) The first coordinate in an ordered pair of real numbers. It indicates the position of a point along the x-axis. Example: The x-coordinate of point P is -4.

y-coordinate (or ordinate) The second coordinate in an ordered pair of real numbers. It indicates the position of a point along the y-axis. Example: The y-coordinate of point P is 2.

Origin The point $(0, 0)$, where the x-axis intersects the y-axis in a coordinate plane.

Linear equation in two variables Any equation that can be written in the form $Ax + By = C$ (A and B not both 0).

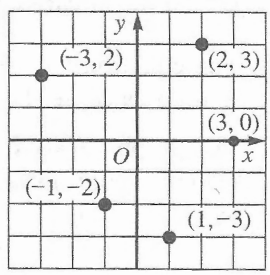
Theorem The graph of every equation of the form $Ax + By = C$ (A and B not both 0) is a line. Conversely, every line in the coordinate plane is the graph of an equation of this form.

Symbol $P(a, b)$ (point P with coordinates (a, b))

Example 1 Graph the ordered pairs $(2, 3)$, $(3, 0)$, $(-3, 2)$, $(1, -3)$, and $(-1, -2)$ in the same xy-coordinate plane.

Solution On the x-axis, the positive direction is to the right, and the negative direction is to the left.

On the y-axis, the positive direction is up, and the negative direction is down.



For each exercise, graph the ordered pairs in the same coordinate plane.

1. $(2, 6)$, $(3, 0)$, $(-2, 3)$, $(4, -5)$, $(0, -4)$

2. $(-1, -3)$, $(-4, 0)$, $(-1, 5)$, $(4, 0)$, $(3, 5)$

3. $(-2, -1)$, $(0, 0)$, $(-1, 2)$, $\left(\frac{5}{2}, -\frac{3}{2}\right)$, $(1, 2)$

4. $(-1, -1)$, $(-2, 2)$, $(3, 3)$, $\left(\frac{3}{2}, \frac{3}{2}\right)$, $(2, -2)$

NAME ______________________ DATE ______________

3–2 Graphs of Linear Equations in Two Variables *(continued)*

Example 2 Graph $2x - 4y = 12$.

Solution Two points are needed to determine a line. The graph crosses the y-axis at a point whose x-coordinate is 0. The graph crosses the x-axis at a point whose y-coordinate is 0. Find these two points and use them to make your graph.

Let $x = 0$.

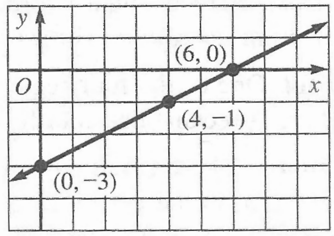
$$2(0) - 4y = 12$$
$$-4y = 12$$
$$y = -3$$

Solution $(0, -3)$

Let $y = 0$.

$$2x - 4(0) = 12$$
$$2x = 12$$
$$x = 6$$

Solution $(6, 0)$



The graph is the line through the points with coordinates $(0, -3)$ and $(6, 0)$.

As a check, note that $(4, -1)$ is a solution of $2x - 4y = 12$ and its graph lies on the line. To avoid errors, it's best to use at least three points to graph a line.

CAUTION When a line passes through the origin, the method used in Example 2 will give you only one solution. In such cases, you will have to let x equal a number besides 0 and solve for y to find another solution.

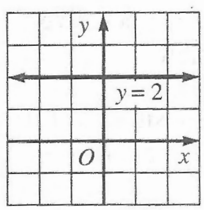
Graph each equation.

5. $x + y = 6$
6. $x - y = 3$
7. $2x + y = 2$
8. $x + 3y = 9$
9. $3x - y = 6$
10. $2x + 7y + 14 = 0$
11. $2x - 3y - 6 = 0$
12. $x + y = 0$
13. $4x - y = 0$
14. $y = 2 + x$
15. $4x - 3y = 0$
16. $4x - 2y - 6 = 0$
17. $2x + 3y = 5$
18. $x + \frac{1}{2}y = 1$
19. $y = \frac{1}{3}x + 2$

Example 3 Graph the equation $y = 2$ in a coordinate plane.

Solution The equation $y = 2$ can be written as $0x + 1y = 2$.

The graph consists of all points having y-coordinate 2 and is therefore a horizontal line.



Note: The graph of $Ax + By = C$ is a horizontal line if $A = 0$, and is a vertical line if $B = 0$.

Graph in a coordinate plane.

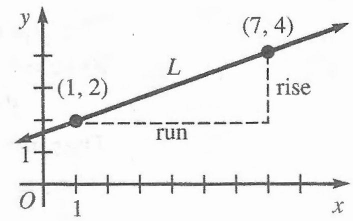
20. $y = -2$
21. $x = 1$
22. $2x + 3 = 0$
23. $2y - 5 = 0$

NAME ______________________ DATE ______________

3–3 The Slope of a Line

Objective: To find the slope of a line and to graph a line given its slope and a point on it.

Vocabulary



Slope of a line L A measure of the "steepness" of the line.
If points (x_1, y_1) and (x_2, y_2) are on line L, and $(x_1 \neq x_2)$,

$$\text{slope of line } L = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example: The slope of line L in the diagram is

$$\frac{\text{rise}}{\text{run}} = \frac{4 - 2}{7 - 1} = \frac{2}{6} = \frac{1}{3}.$$

Coefficient The constant (or numerical) factor of a term.
Example: The coefficient of the term $5x^3$ is 5.

Symbol (x_1, y_1) (read "x one, y one" or "x sub one, y sub one")

CAUTION A vertical line has no slope. A horizontal line has slope 0. Having slope 0 is not the same as having no slope.

Example 1 Find the slope of the line containing the given points.

a. $(-3, 2)$ and $(5, -1)$ **b.** $(6, -1)$ and $(-3, -1)$

Solution **a.** slope $= \dfrac{-1 - 2}{5 - (-3)} = -\dfrac{3}{8}$ **b.** slope $= \dfrac{-1 - (-1)}{-3 - 6} = \dfrac{0}{-9} = 0$

$\therefore$ the line is horizontal.

Find the slope of the line containing the given points. If the line has no slope, write "vertical."

1. $(2, 3), (5, -2)$

2. $(3, 1), (0, 1)$

3. $(4, -2), (-2, 1)$

4. $(-5, 2), (-5, 4)$

5. $(4, 2), (-1, -3)$

6. $(-2, 3), (4, -1)$

7. $(-2, -4), (-2, 4)$

8. $\left(\frac{2}{3}, 3\right), \left(\frac{5}{3}, 7\right)$

9. $\left(\frac{1}{3}, -5\right), (0, -4)$

10. $(0.5, 1), (0.75, -2)$

11. $(r, s), (-r, -s)$ $(r \neq 0)$

12. $(-r, -s), (-s, -r)$ $(r \neq s)$

Theorem The slope of the line $Ax + By = C$ $(B \neq 0)$ is $-\dfrac{A}{B}$.
It follows from this theorem that the slope of a line is the coefficient of the x-term when the equation of the line is solved for y: $y = -\dfrac{A}{B}x + \dfrac{C}{B}$.

3–3 The Slope of a Line *(continued)*

Example 2 Find the slope of the line $7x - 4y = 20$.

Solution

Method 1:

The equation is written in the form $Ax + By = C$, where $A = 7$ and $B = -4$.

Therefore, the slope is

$$-\frac{A}{B} = -\frac{7}{-4} = \frac{7}{4}.$$

Method 2:

Solve the given equation for y.

$$7x - 4y = 20$$
$$-4y = -7x + 20$$
$$y = \frac{7}{4}x - 5$$

The slope is the coefficient of x, $\frac{7}{4}$.

Find the slope of each line.

13. $x + y = 3$ **14.** $x - y - 2 = 0$ **15.** $3x + 2y = 4$ **16.** $3y + 2 = 9x$

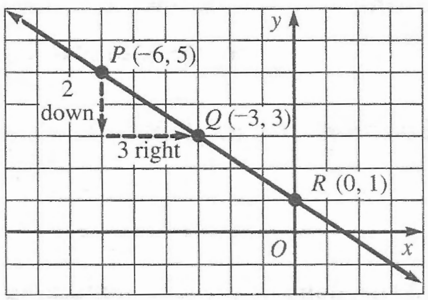
17. $-x = y - 6$ **18.** $3(-2x + 1) = 8y$ **19.** $\frac{1}{3}x - \frac{1}{2}y = 1$ **20.** $\frac{x}{4} - \frac{y}{-6} = 1$

Example 3 Graph the line through the point $P(-6, 5)$ having slope $m = -\frac{2}{3}$.

Solution To graph the line, start at a known point and use the slope to find a second point.

$$m = \frac{\text{rise}}{\text{run}} = -\frac{2}{3} = \frac{-2}{3}$$

You will start at $P(-6, 5)$ and move 2 units *down* and 3 units *to the right* to reach a second point $Q(-3, 3)$. Then you can draw the line through P and Q. To find other points on the line, repeat this process. Another point is $R(0, 1)$.



Graph the line through point P having slope m. Find the coordinates of two other points on the line.

21. $P(-1, -2)$, $m = 3$ **22.** $P(0, 0)$, $m = -2$ **23.** $P(1, 2)$, $m = \frac{1}{2}$

24. $P(3, -1)$, $m = -\frac{3}{2}$ **25.** $P(-3, 2)$, $m = 0$ **26.** $P(2, 2)$, no slope

Mixed Review Exercises

Complete the ordered pair to form a solution of the given equation.

1. $x + 2y = 3$; (_?_, 1) **2.** $x - 2y = 4$; (_?_, −5) **3.** $4x - 5y = 2$; (3, _?_)

Graph each equation.

4. $x - y = 3$ **5.** $4x - 5y = 8$ **6.** $y = x$ **7.** $y - 3 = 0$ **8.** $y = -\frac{1}{3}x - 2$

NAME ______________________ DATE ______________

3–4 Finding an Equation of a Line

Objective: To find an equation of a line given its slope and a point on the line, or two points, or its slope and the y-intercept.

Vocabulary

Standard form (of a linear equation in two variables) The equation of a line written as $Ax + By = C$, where A, B, and C are integers, and A and B are not both zero.

Point-slope form The equation of a line written as $y - y_1 = m(x - x_1)$ where m is the slope and (x_1, y_1) is a point on the line.

Example 1 Find an equation in standard form of the line containing the point $(1, -5)$ and having slope $-\frac{2}{3}$.

Solution Use the point-slope form with $(x_1, y_1) = (1, -5)$ and $m = -\frac{2}{3}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-5) &= -\frac{2}{3}(x - 1) \\ 3(y + 5) &= -2(x - 1) \\ 3y + 15 &= -2x + 2 \\ 2x + 3y &= -13 \end{aligned}$$

{Multiply both sides by 3 to clear the fractions.

Check: $2(1) + 3(-5) = -13$; $m = -\frac{A}{B} = -\frac{2}{3}$ √

Find an equation in standard form of the line containing point P and having slope m.

1. $P(4, 1)$, $m = -1$ **2.** $P(3, 0)$, $m = -2$ **3.** $P(-2, 1)$, $m = 0$ **4.** $P(-1, -3)$, $m = 3$

5. $P(-2, 2)$, $m = \frac{1}{4}$ **6.** $P(2, -1)$, $m = \frac{1}{2}$ **7.** $P(3, 4)$, $m = -\frac{5}{4}$ **8.** $P(-1, 5)$, $m = 0.2$

Example 2 Find an equation in standard form of the line containing $(2, -3)$ and $(-1, 2)$.

Solution Find the slope of the line: $m = \frac{2 - (-3)}{-1 - 2} = \frac{5}{-3} = -\frac{5}{3}$

Next use the point-slope form with either point.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-3) &= -\frac{5}{3}(x - 2) \\ 3(y + 3) &= -5(x - 2) \\ 3y + 9 &= -5x + 10 \\ 5x + 3y &= 1 \end{aligned}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= -\frac{5}{3}(x - (-1)) \\ 3(y - 2) &= -5(x + 1) \\ 3y - 6 &= -5x - 5 \\ 5x + 3y &= 1 \end{aligned}$$

Find an equation in standard form of the line containing the given points.

9. $(4, 6)$, $(0, 0)$ **10.** $(-1, 3)$, $(3, -1)$ **11.** $(1, -2)$, $(-2, -1)$ **12.** $(7, 2)$, $(-1, -2)$

13. $(-3, 1)$, $(-4, 1)$ **14.** $(4, -3)$, $(4, 5)$ **15.** $\left(\frac{1}{4}, \frac{3}{2}\right)$, $\left(\frac{3}{4}, 1\right)$ **16.** $\left(\frac{1}{3}, -\frac{1}{2}\right)$, $\left(\frac{4}{3}, \frac{3}{2}\right)$

NAME ______________________________ DATE ______________

3–4 Finding an Equation of a Line *(continued)*

Vocabulary

y-intercept The y-coordinate of the point where a line (or curve) intersects the y-axis.

x-intercept The x-coordinate of the point where a line (or curve) intersects the x-axis.

Slope-intercept form The equation of a line written as $y = mx + b$ where m is the slope and b is the y-intercept.

Theorem Let L_1 and L_2 be two different lines, with slopes m_1 and m_2.

1. L_1 and L_2 are *parallel* if and only if $m_1 = m_2$.
2. L_1 and L_2 are *perpendicular* if and only if $m_1 m_2 = -1$. (m_1 and m_2 are *negative reciprocals*.)

Example 3 Find an equation in standard form of the line having slope $-\frac{3}{4}$ and y-intercept 2.

Solution Use $y = mx + b$ with $m = -\frac{3}{4}$ and $b = 2$.

$$y = -\frac{3}{4}x + 2$$
$$4y = -3x + 8$$
$$3x + 4y = 8$$

Multiply both sides by 4 to clear the fraction.

Find an equation in standard form of the line having slope m and y-intercept b.

17. $m = -1, b = 3$ **18.** $m = 2, b = -4$ **19.** $m = -\frac{3}{2}, b = \frac{1}{2}$ **20.** $m = 0.8, b = 0.6$

Example 4 Find equations in standard form of the lines through point $P(-1, 3)$ that are (a) parallel to and (b) perpendicular to the line $x - 3y = 6$.

Solution Solve the equation for y to find the slope:

$x - 3y = 6;\ y = \frac{1}{3}x - 2$ $\quad\therefore$ the slope of the line is $\frac{1}{3}$.

a. A line parallel to $x - 3y = 6$ has the same slope, $\frac{1}{3}$. Use the point-slope form:

$$y - 3 = \frac{1}{3}(x + 1)$$
$$3y - 9 = x + 1$$
$$x - 3y = -10$$

b. The slope of a line perpendicular to $x - 3y = 6$ is the negative reciprocal of $\frac{1}{3}$, or -3. Use the point-slope form:

$$y - 3 = -3(x + 1)$$
$$y - 3 = -3x - 3$$
$$3x + y = 0$$

Find equations in standard form of the lines through point P that are (a) parallel to and (b) perpendicular to line L.

21. $P(0, -5);\ L: y = \frac{3}{4}x + 2$

22. $P(3, 0);\ L: x - 4y = 4$

23. $P(2, -3);\ L: 2x + 7y = 14$

24. $P(-5, -1);\ L: x + 4 = 0$

NAME ______________________ DATE ______________

3–5 Systems of Linear Equations in Two Variables

Objective: To solve systems of linear equations in two variables.

Vocabulary

System of linear equations A set of linear equations in the same two variables.

Transformations that produce equivalent systems (systems with the same solution set)

1. Replacing an equation by an equivalent equation.
2. Substituting for one variable in any equation an equivalent expression for that variable obtained from another equation in the system.
3. Replacing any equation by the sum of that equation and another equation in the system.

Linear combination The equation that results when two linear equations are added.

Example 1 Solve these systems:

a. $3x + 4y = 1$
$5x - 3y = 21$

b. $2x + y = 0$
$x - 3y = -7$

Solution

a. Find a linear combination that eliminates y.

1. Multiply the first equation by 3 and the second equation by 4 so that the coefficients of y will be *opposites*.

$$\begin{aligned} 9x + 12y &= 3 \\ 20x - 12y &= 84 \end{aligned}$$

2. Add the equations in Step 1 and solve the resulting equation for x.

$$\begin{aligned} 29x + 0y &= 87 \\ 29x &= 87 \\ x &= 3 \end{aligned}$$

3. Substitute 3 for x in either of the *original* equations to find y.

$$\begin{aligned} 3(3) + 4y &= 1 \\ 9 + 4y &= 1 \\ 4y &= -8 \\ y &= -2 \end{aligned}$$

4. Check that $(3, -2)$ satisfies both of the original equations.

$$\begin{aligned} 3(3) + 4(-2) &= 1 \ \checkmark \\ 5(3) - 3(-2) &= 21 \ \checkmark \end{aligned}$$

$\therefore$ the solution is $(3, -2)$.

b. Use the substitution method.

1. Express y in terms of x in the first equation (since the coefficient of y is 1).

$$\begin{aligned} 2x + y &= 0 \\ y &= -2x \end{aligned}$$

2. Substitute $-2x$ for y in the second equation. Solve for x.

$$\begin{aligned} x - 3y &= -7 \\ x - 3(-2x) &= -7 \\ x + 6x &= -7 \\ 7x &= -7 \\ x &= -1 \end{aligned}$$

3. Substitute -1 for x in either of the original equations to find y.

$$\begin{aligned} -1 - 3y &= -7 \\ -3y &= -6 \\ y &= 2 \end{aligned}$$

4. The check is left for you.

$\therefore$ the solution is $(-1, 2)$.

NAME ______________________ DATE ______________

3-5 Systems of Linear Equations in Two Variables *(continued)*

CAUTION The equations of a system are **consistent** and have exactly one solution, as in Example 1, when their graphs are *intersecting lines*. NOT all systems have one solution.

Example 2 Solve these systems: **a.** $2x + y = -1 - 2y$, $2x - 9y = 3 + 8x$ **b.** $2y = x + 3$, $2y - x - 6 = 0$

Solution **a.** Transform each equation into standard form. Use a linear combination.

$$\left.\begin{aligned} 2x + y &= -1 - 2y \\ 2x - 9y &= 3 + 8x \end{aligned}\right\} \longrightarrow \begin{aligned} 2x + 3y &= -1 \xrightarrow{\times 3} & 6x + 9y &= -3 \\ -6x - 9y &= 3 & -6x - 9y &= 3 \\ & & 0 &= 0 \end{aligned}$$

Since $0 = 0$ is an identity, all solutions of one equation are solutions of the other. Thus, the solution set is $\{(x, y): 2x + 3y = -1\}$. The equations are consistent and **dependent** and their graphs *coincide*.

b. Solve the first equation for x, and use the substitution method.

$$\begin{aligned} 2y &= x + 3 & 2y - x - 6 &= 0 \\ x &= 2y - 3 \xrightarrow{\text{substitute}} & 2y - (2y - 3) - 6 &= 0 \\ & & 2y - 2y + 3 - 6 &= 0 \\ & & -3 &= 0 \longleftarrow \text{False!} \end{aligned}$$

Since $-3 = 0$ is false, the system has no solution. The equations are **inconsistent** and their graphs are *parallel lines*.

Solve each system. If the system has an infinite solution set, specify it and give three solutions. If the system has no solution, say so.

1. $2x + 5y = 41$; $2x + y = 13$
2. $2p - q = -1$; $3p - 4q = 6$
3. $8x + 3y = 23$; $4x - 5y = 5$
4. $3x + 2y = 22$; $2x - 3y = 6$
5. $5x - 7y = 54$; $2x - 3y = 22$
6. $3d + 7c = 42$; $4d + 5c = 30$
7. $3u + 4v = 9$; $5u - 8v = 4$
8. $2x - 5y = 2$; $7x + \frac{1}{2}y = -1$
9. $8n = 6m - 3$; $9m = 12n + 5$
10. $4x + 3y = x + 6$; $x + 3y - 2 = 2y$
11. $3(5 - x) = y$; $5(3 - x) = -2y + 1$
12. $2b = 2a - b + 4$; $3a = a + 3b + 4$

Mixed Review Exercises

For the line containing the given points, find (a) the slope and (b) an equation in standard form.

1. $(3, 2), (-1, -6)$
2. $(0, 3), (-1, 2)$
3. $(1, -2), (-1, -2)$
4. $(0, -2), (2, -1)$

Find the slope and *y*-intercept of each line.

5. $2x - 2y = 6$
6. $2x + 3y = 9$
7. $-x + 2y = 4$
8. $y + 7 = 0$

NAME ______________________ DATE ______________

3–6 Problem Solving: Using Systems

Objective: To use systems of equations to solve problems.

Example 1 An algebra test contains 38 problems. Some of the problems are worth 2 points each. The rest are worth 3 points each. A perfect score is 100 points. How many problems are worth 2 points?

Solution

Step 1 The problem asks for the number of 2-point problems on the test.

Step 2 Let a = the number of 2-point problems on the test.
Let b = the number of 3-point problems on the test.

Step 3 Set up a system of two equations.
Total number of problems on the test is 38. ⟶ $a + b = 38$
Total point value of the test is 100. ⟶ $2a + 3b = 100$

Step 4 Solve the system. Multiply the first equation by -2 to eliminate the a's, and solve for b. Then substitute $b = 24$ in either of the original equations and solve for a.

$$\begin{aligned} -2a - 2b &= -76 \\ 2a + 3b &= 100 \\ \hline b &= 24 \end{aligned}$$

$$\begin{aligned} a + 24 &= 38 \\ a &= 14 \end{aligned}$$

There are 14 problems worth 2 points each and 24 problems worth 3 points each.

Step 5 Check your answer against the original problem.
The test contains 38 problems: $14 + 24 = 38$ √
A perfect score is 100 points: $2(14) + 3(24) = 28 + 72 = 100$ √
∴ 14 problems on the test are worth 2 points.

Solve.

1. Tickets for the Senior Prom cost \$25 for a single ticket and \$40 for a couple. Ticket sales totaled \$3800 and 110 tickets were sold. How many tickets of each type were sold?

2. A cashier had to give Sarah \$3.45 in change but he had only quarters and dimes in the cash register. If he gave her 15 coins, how many dimes did she receive?

Vocabulary

Air speed The speed of an aircraft in still air.

Wind speed The speed of the wind.

Tail wind A wind blowing in the same direction as the path of the aircraft.

Head wind A wind blowing in the direction opposite to the path of the aircraft.

Ground speed The speed of the aircraft relative to the ground.

With a tail wind: ground speed = air speed + wind speed.
With a head wind: ground speed = air speed − wind speed.

3–6 Problem Solving: Using Systems *(continued)*

Example 2 With a tail wind, a plane traveled 840 miles in 3 hours. With the same wind, as a head wind, the return trip took 30 min longer. Find the plane's air speed and the wind speed.

Solution

Step 1 The problem asks for the plane's air speed and the speed of the wind.

Step 2 Let p = the air speed of the plane in miles per hour.
Let w = the wind speed in miles per hour.

Step 3 Use the formula rate × time = distance to construct a table.

	Rate (ground speed in mi/h) ×	Time (hours) =	Distance (miles)
With tail wind	$p + w$	3	$3(p + w)$
With head wind	$p - w$	$3\frac{1}{2}$ or $\frac{7}{2}$	$\frac{7}{2}(p - w)$

The distance with a tail wind is 840 mi. ⟶ $3(p + w) = 840$

The distance with a head wind is 840 mi. ⟶ $\frac{7}{2}(p - w) = 840$

Step 4 Solve the system.

$3(p + w) = 840$ is equivalent to: $p + w = 280$

$\frac{7}{2}(p - w) = 840$ is equivalent to: $p - w = 240$

$$2p = 520$$
$$p = 260$$
$$3(260 + w) = 840$$
$$260 + w = 280$$
$$w = 20$$

The solution is (260, 20).

Step 5 The check is left for you.

∴ the air speed of the plane is 260 mi/h and the speed of the wind is 20 mi/h.

Solve.

3. With a head wind, a plane traveled 840 miles northward in 2 hours. With the same wind as a tail wind, the return trip southward took 1 hour and 45 minutes. Find the plane's air speed and the wind speed.

4. With a tail wind, a small plane traveled 420 miles in 1 hour and 30 minutes. The return trip against the same wind took 30 minutes longer. Find the wind speed and the air speed of the plane.

NAME ______________________ DATE ______________

3–7 Linear Inequalities in Two Variables

Objective: To graph linear inequalities in two variables and systems of such inequalities.

Vocabulary

Linear inequality in two variables A sentence obtained by replacing the equals sign in a linear equation by an inequality symbol ($>$, $<$, $\geq$, or $\leq$). Examples: $4x - 7y \geq 5$, and $x < 9y$.

Solution of an inequality in two variables An ordered pair of numbers that satisfies the inequality, or makes it true. Example: (5, 9), (0, 0), and (−1, 2) are solutions of the inequality $y \leq 2x + 4$.

Associated equation The linear equation obtained by replacing the inequality symbol in a linear inequality by an equals sign. Example: The associated equation of the inequality $y > 2x + 5$ is $y = 2x + 5$.

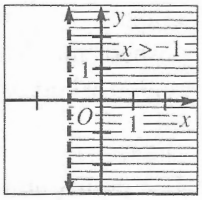
Open half-plane The set of all points on one side of a given line (the *boundary*). The boundary appears as a dashed line because it is *not* included in the graph.

Closed half-plane An open half-plane together with its boundary. The boundary appears as a solid line because it *is* included in the graph.

System of linear inequalities A set of linear inequalities in the same two variables. The graph of such a system is the set of points satisfying *all* of its inequalities.

Example 1 Graph $x > -1$ in a coordinate plane.

Solution The graph of the associated equation $x = -1$ is the boundary. Since x is greater than -1, the boundary line is dashed. A point belongs to the graph of the inequality if and only if its x-coordinate is greater than -1. This means that you must shade the points to the *right* of the boundary line.

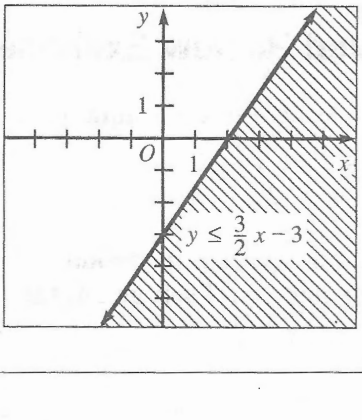


Example 2 Graph $3x - 2y \geq 6$.

Solution Solve the inequality for y.

$$3x - 2y \geq 6$$
$$-2y \geq -3x + 6$$
$$y \leq \frac{3}{2}x - 3$$

Graph the associated equation $y = \frac{3}{2}x - 3$ as a solid line. The graph of the inequality is a closed half-plane that includes this boundary and the points *below* it.



NAME ______________________ DATE ______________

3–7 *Linear Inequalities in Two Variables* (continued)

CAUTION Don't assume that you should shade below for an inequality having the symbol $<$, and above for the symbol $>$. This is only true when an inequality has been solved for y in terms of x. A safer way to determine which side of the boundary to shade is to test a point on either side of it. The origin, (0, 0), is a convenient point to use. If the point you test satisfies the inequality, you shade the region containing that point. If not, shade the opposite region.

Graph each inequality in a coordinate plane.

1. $y \geq 1$ **2.** $x + 2 > 0$ **3.** $x - y \geq 0$ **4.** $2x < y$

5. $x + y < 2$ **6.** $3x - y \leq 1$ **7.** $4x - y \geq 2$ **8.** $3x + 4y > 8$

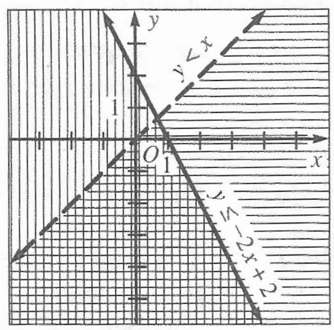
9. $5x - 2y \leq 6$ **10.** $2x < -3(2 - y)$ **11.** $y \leq -\frac{1}{2}(5x - 4)$ **12.** $4(x + 2) > 3y + 2$

Example 3 Graph this system:
$$2x + y \leq 2$$
$$x - y > 0$$

Solution As in Example 2, solve each inequality for y.

$2x + y \leq 2$ $\quad y \leq -2x + 2$

$x - y > 0$ $\quad y < x$



Graph $y \leq -2x + 2$, showing the boundary as a solid line. In the same coordinate plane, graph $y < x$, showing the boundary as a dashed line. The graph of the system is the region where the two graphs overlap.

Note: The points on the solid boundary are part of the solution set. The points on the dashed boundary are not.

Graph each system of inequalities.

13. $y \geq 1$, $y < x + 1$

14. $x + y < 0$, $2x - y > 0$

15. $y < 2x - 1$, $1 - y \geq 0$

16. $2x + y \geq 3$, $2x - y \leq 3$

17. $5x - 2y < 10$, $2x - 5y > 10$

Mixed Review Exercises

Each system has a unique solution; find it.

1. $4x - y = 2$, $x - 2y = 4$

2. $2x - 3y = -1$, $x + y = 7$

3. $-2x + y = 0$, $3x + 2y = -7$

Find an equation in standard form of the line through P having the given slope m. Then graph the line.

4. $P(-1, 4)$, $m = 2$ **5.** $P(3, 5)$, $m = 0$ **6.** $P(0, 0)$, $m = -2$

7. $P(-5, -1)$, $m = \frac{1}{3}$ **8.** $P(-2, 3)$, $m = -\frac{4}{5}$ **9.** $P(2, 1)$, no slope

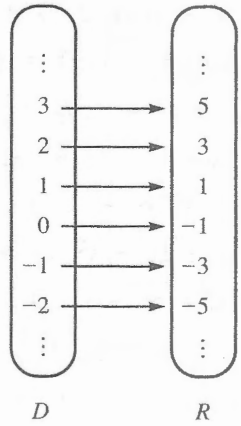
NAME ____________________ DATE ____________

3–8 Functions

Objective: To find values of functions and to graph functions.

Vocabulary

Mapping diagram A diagram such as the one on the right that pictures a correspondence between two sets.



Function A correspondence between two sets, D and R, that assigns to each member of D exactly one member of R. Functions are sometimes specified by a rule. The rule for the function in the diagram is "double the number and subtract 1."

Domain The domain of a function is the set D. In the diagram, $D = \{\text{integers}\}$.

Range The range of a function is the set of all members of R assigned to at least one member of D. In the diagram, $R = \{\text{odd integers}\}$.

Values of a function Members of the range of the function.

Graph of a function The set of all points (x, y) such that x is in the domain of the function and the rule of the function assigns y to x.

Arrow notation Notation used to describe the rule of a function.
Example: The arrow notation for the function in the diagram is $f\colon x \to 2x - 1$.
This is read "f, the function that assigns $2x - 1$ to the number x."

Functional notation Another type of notation used to describe the rule of a function.
Example: The functional notation for the function in the diagram is $f(x) = 2x - 1$.
$f(3) = 5$ is read "the value of the function f at 3 is 5" or "f of 3 is 5."

Example 1 Given $f\colon x \to x^2 - 3x$ with domain $D = \{-1, 0, 1, 2, 3\}$.

a. Find the range of f. **b.** Graph the function.

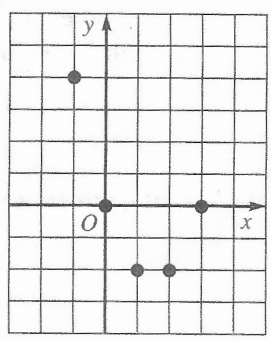
Solution **a.** Make a table showing the numbers that the rule of f assigns to each member of the domain.

x	$x^2 - 3x$	(x, y)
-1	$(-1)^2 - 3(-1) = 4$	$(-1, 4)$
0	$0^2 - 3(0) = 0$	$(0, 0)$
1	$1^2 - 3(1) = -2$	$(1, -2)$
2	$2^2 - 3(2) = -2$	$(2, -2)$
3	$3^2 - 3(3) = 0$	$(3, 0)$

The range R is $\{4, 0, -2\}$.

Note: You list each member of the range only once.

b. Graph the ordered pairs from the table.



NAME ______________________ DATE ______________

3–8 Functions *(continued)*

Find the range of each function given the domain D and the rule of the function.

1. $F: x \to 3x - 2;\ D = \{-1, 0, 1, 2\}$
2. $G: x \to 4x + 2;\ D = \{0, 1, 2, 3\}$
3. $g: x \to 3 - x;\ D = \{0, 1, 2, 3\}$
4. $f: x \to x^2 + 1;\ D = \{-2, -1, 0, 1\}$
5. $h: x \to 4 - 4x + x^2;\ D = \{0, 1, 2, 3, 4\}$
6. $\phi: x \to 2x^2 - 3x + 1;\ D = \{-1, 0, 1, 2\}$
7. $g: x \to 2x - x^2;\ D = \{-2, -1, 0, 1, 2\}$
8. $H: x \to x^3 - x^2;\ D = \{-1, 0, 1, 2\}$
9. $h: x \to |x - 2|;\ D = \{0, 1, 2, 3, 4\}$
10. $f: x \to |x| - x;\ D = \{-2, -1, 0, 1, 2\}$

11–20. Graph the functions in Exercises 1–10.

Example 2 Give the domain of each function.

a. $f(x) = \dfrac{3}{x^2(x - 5)}$ **b.** $g(x) = \dfrac{5}{x^2 + 1}$ **c.** $h(x) = \sqrt{x + 1}$

Solution Begin with the set of real numbers. Then eliminate any of these that don't produce a real number when substituted into the rule for the function.

a. The expression $\dfrac{3}{x^2(x - 5)}$ is not defined when the denominator equals zero, in other words, when $x = 0$ or $x = 5$.

$\therefore D = \{\text{all real numbers except } 0 \text{ and } 5\}$

b. There is no real number that will make the denominator zero, since $x^2 > 0$ for all real numbers.

$\therefore D = \{\text{real numbers}\}$

c. The expression $\sqrt{x + 1}$ is defined only when $x + 1 \geq 0$, otherwise there would be a negative number inside the square root symbol.

$\therefore D = \{x: x \geq -1\}$

Give the domain of each function.

21. $f(x) = x + 7$
22. $g(x) = \dfrac{3}{x + 1}$
23. $h(x) = \dfrac{3}{(x - 2)(x + 2)}$
24. $F(x) = \dfrac{3}{x^2 - 9}$
25. $G(x) = \sqrt{2x - 1}$
26. $H(x) = \sqrt{x^2 + 6}$

NAME ______________________________ DATE ______________

3–9 Linear Functions

Objective: To find equations of linear functions and to apply properties of linear functions.

Vocabulary

Linear function A function f that can be defined by $f(x) = mx + b$ where x, m, and b are any real numbers. The graph of f is the graph of $y = mx + b$, a line with slope m and y-intercept b.

Constant function A linear function where $m = 0$. Its equation is $f(x) = b$.

Rate of change of $f(x)$ A number that tells you how much $f(x)$ changes for any given change in x. In a linear function $f(x) = mx + b$, the constant m is the rate of change. The rate of change can be computed using the formula:

$$\text{rate of change } m = \frac{\text{change in } f(x)}{\text{change in } x}$$

Example: If $f(x)$ decreases by 3 when x increases by 5, then $m = \frac{-3}{5} = -\frac{3}{5}$.

Example 1 Find equations of the linear functions g and h using the given information.

a. $m = -2$ and $g(3) = 5$ **b.** $h(1) = 4$ and $h(-2) = 10$

Solution

a. Since g is linear and $m = -2$, substitute -2 for m in $g(x) = mx + b$. → $g(x) = -2x + b$

$g(3) = 5$ means that $g(x) = 5$ when $x = 3$. Substitute and solve for b. →

$$5 = -2(3) + b$$
$$5 = -6 + b$$
$$11 = b$$

$\therefore\ g(x) = -2x + 11$

b. Since h is linear, $h(x) = mx + b$. First you need to find m.

$$m = \frac{\text{change in } h(x)}{\text{change in } x} = \frac{h(1) - h(-2)}{1 - (-2)} = \frac{4 - 10}{1 + 2} = \frac{-6}{3} = -2$$

Substitute -2 for m in $h(x) = mx + b$. → $h(x) = -2x + b$

To find the value of b, use either $h(1) = 4$ or $h(-2) = 10$. →

$$4 = -2(1) + b$$
$$4 = -2 + b$$
$$6 = b$$

$\therefore\ h(x) = -2x + 6$

Find an equation of the linear function f using the given information.

1. $m = 4,\ b = 1$

2. $m = -2,\ b = 3$

3. $f(0) = 1$, slope of graph $= -3$

4. $f(0) = 3$, slope of graph $= -\frac{1}{2}$

5. $f(0) = -2$; $f(x)$ increases by 12 when x increases by 3

6. $f(0) = -1$; $f(x)$ decreases by 2 when x increases by 3

7. $m = 6,\ f(1) = 5$

8. $m = -\frac{2}{3},\ f(3) = 2$

9. $f(1) = 3,\ f(-2) = -6$

10. $f(-3) = 1,\ f(2) = 2$

11. $f(4) = -5,\ f(-3) = 3$

12. $f(-3) = 5,\ f(2) = 5$

NAME ______________________________ DATE ______________

3-9 Linear Functions *(continued)*

Example 2 The prom committee asked a local restaurant to cater the senior prom. The restaurant charges \$1700 for a buffet for 100 people. It charges \$2450 for a buffet for 150 people.

a. Write a linear function to describe this situation.
b. Find the cost of a buffet for 175 people.

Solution

a. Let $f(n)$ = the cost of a buffet for n people.
Since f is linear, $f(n) = mn + b$.

To find the values of m and b use the facts that $f(100) = 1700$ and $f(150) = 2450$ to write a system of linear equations:

$$f(100) = 100m + b = 1700$$
$$f(150) = 150m + b = 2450$$

Multiply the first equation by -1 and solve for m.

$$\begin{aligned} -100m - b &= -1700 \\ 150m + b &= 2450 \\ \hline 50m &= 750 \\ m &= 15 \end{aligned}$$

Substitute m in one of the original equations and solve for b.

$$\begin{aligned} 100(15) + b &= 1700 \\ 1500 + b &= 1700 \\ b &= 200 \end{aligned}$$

$\therefore f(n) = 15n + 200$

b. Substitute $n = 175$ in the equation obtained in part (a).

$f(175) = 15(175) + 200 = 2625 + 200 = 2825$

$\therefore$ the cost of a buffet for 175 people is \$2825.

Solve the following problems. Assume that each situation can be described by a linear function.

13. An electrician charged \$90 for a two-hour job and \$140 for a four-hour job. At this rate, how much would be charged for an eight-hour job?

14. The relationship between hours spent studying and the average grade on an algebra exam was found to be a linear function. The average grade for students who spent 1 hour studying was 76, and it was 88 for students who spent 3 hours studying.

a. What was the average grade for students who spent 4 hours studying?
b. How many hours of studying would have resulted in an average grade of 80?

Mixed Review Exercises

Graph each open sentence in a coordinate plane.

1. $2x - y > 3$ **2.** $y = -x$ **3.** $x - 3 \leq 0$ **4.** $x - 3y \leq -4$

Find the range of each function with domain D.

5. $g: x \to 3 - 2x^2, D = \{-2, -1, 0\}$ **6.** $f(x) = |x| + 3, D = \{-2, -1, 1\}$

NAME ______________________ DATE ______________

3–10 Relations

Objective: To graph relations and to determine when relations are also functions.

Vocabulary

Relation Any set of ordered pairs. The set of first coordinates in the ordered pairs is the *domain*; the set of second coordinates is the *range*.
Example: The set $\{(1, 2), (3, 4), (3, 5)\}$ is a relation, where $D = \{1, 3\}$ and $R = \{2, 4, 5\}$.

Function A relation in which different ordered pairs have different *first* coordinates.
Example: The set $\{(0, -1), (2, 3), (5, 9)\}$ is a function. The set $\{(1, 2), (3, 4), (3, 5)\}$ is not.

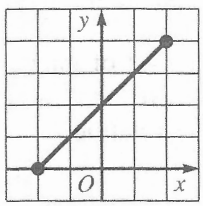
Vertical-line test A test to determine whether a given relation is a function. This test says that a relation is a function if and only if no vertical line can intersect its graph more than once.

Symbols $\{(x, y): |x| + |y| = 1\}$ (An example of set notation for ordered pairs. It reads "the set of all ordered pairs (x, y) such that $|x| + |y| = 1$.")

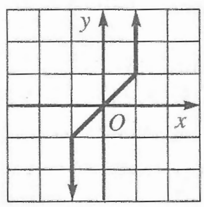
CAUTION A relation can contain different ordered pairs with the same x-value or y-value. A function can contain different ordered pairs with the same y-value, but *not* with the same x-value. (The latter are points on a vertical line.)

Example 1 Give the domain and range of each relation, and tell whether it's a function.

a.



b.



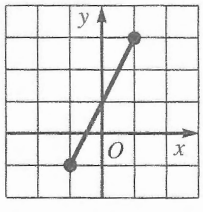
Solution

a. The domain is $\{x: -2 \leq x \leq 2\}$, since the graph goes from -2 to 2 with respect to the x-axis. The range is $\{y: 0 \leq y \leq 4\}$, since the graph goes from 0 to 4 with respect to the y-axis. The relation *is* a function, since each value of x in the graph is paired with exactly one value of y.

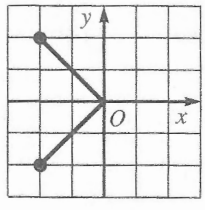
b. The domain is $\{x: -1 \leq x \leq 1\}$, since the graph goes from -1 to 1 with respect to the x-axis. The range is {real numbers}, since the graph spans the entire y-axis. The relation is *not* a function, since it includes different ordered pairs with the same first coordinate, like (1, 1) and (1, 2).

Give the domain and range of each relation graphed below. Is the relation a function?

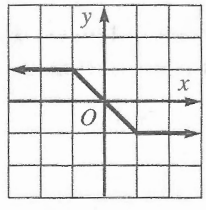
1.



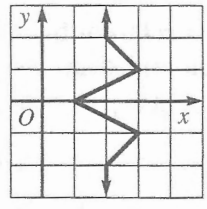
2.



3.



4.

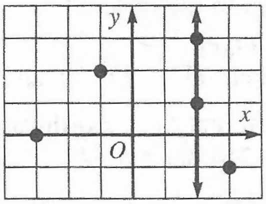


NAME ______________________ DATE ______________

3–10 Relations *(continued)*

Example 2 Graph the relation $\{(2, 1), (3, -1), (2, 3), (-1, 2), (-3, 0)\}$, and tell whether it is a function. If it is not a function, draw a vertical line that intersects the graph more than once.

Solution



The relation is not a function. The vertical line $x = 2$ intersects the graph at two points, (2, 1), and (2, 3).

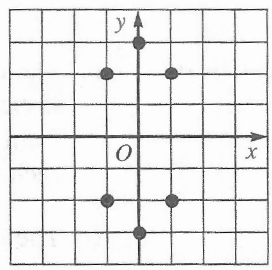
Graph each relation. Then tell whether it is a function. If it is not a function, draw a vertical line that intersects the graph more than once.

5. $\{(2, 1), (0, 0), (2, 2)\}$
6. $\{(-2, 1), (0, 1), (2, 1)\}$
7. $\{(3, 1), (2, -1), (0, 0), (-1, 2)\}$
8. $\{(-2, 2), (-1, 2), (0, 1), (-2, 1)\}$
9. $\{(-1, 2), (3, -1), (2, -2), (-1, -2)\}$
10. $\{(1, 3), (3, -1), (1, -1), (-2, -1)\}$

Example 3 In the relation $\{(x, y): |x| + |y| = 3 \text{ and } |x| \leq 1\}$, x and y are integers. Find the domain D of the relation and draw its graph. Is the relation a function?

Solution Since x is an integer and $|x| \leq 1$, the domain is the set of integers within one unit of the origin, so $D = \{-1, 0, 1\}$. Use a chart to find the ordered pairs for the graph. It will be easier to use the equivalent relation $|y| = 3 - |x|$.

x	$\|y\| = 3 - \|x\|$	y
-1	$\|y\| = 3 - \|-1\| = 2$	2 or -2
0	$\|y\| = 3 - \|0\| = 3$	3 or -3
1	$\|y\| = 3 - \|1\| = 2$	2 or -2



Since a vertical line, such as $x = -1$, intersects the graph at more than one point, the relation is not a function.

CAUTION To find the domain of a relation that is defined by a conjunction, as in Example 3, be sure to consider *both* open sentences. For example, if x is an integer, the domain of $\{x: |x| \leq 1 \text{ and } |y| = x\}$ cannot include -1 since $|y|$ is always nonnegative.

In each relation below, x and y are integers. Find the domain of the relation and draw its graph. Is the relation a function?

11. $\{(x, y): |x| + |y| = 2 \text{ and } |x| \leq 1\}$
12. $\{(x, y): |y| = x \text{ and } x \leq 2\}$
13. $\{(x, y): |x| = y \text{ and } y \leq 2\}$
14. $\{(x, y): |x| = |y| \text{ and } |x| \leq 3\}$
15. $\{(x, y): |x + y| = 0 \text{ and } |x| \leq 3\}$
16. $\{(x, y): |y| + |x| \leq 2 \text{ and } |y| \leq 1\}$