Midpoint Rule

> *restart*

> *with*(*Student*[*Calculus1*]) :

Example 1. Use the Midpoint Rule to approximate the integral $\int_0^{\pi} \sin(x)\, dx$.

> $f := x \to \sin(x)$

$$f := x \to \sin(x) \quad (1)$$

>

> $a := 0;\, b := \pi$

$$a := 0$$
$$b := \pi \quad (2)$$

>

Approximate the integral by partitioning $[0, \pi]$ into 4 subintervals of equal width:

> $N := 4$

$$N := 4 \quad (3)$$

> $\Delta x := \frac{(b-a)}{N}$

$$\Delta x := \frac{1}{4}\pi \quad (4)$$

> $Sum\left(f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right) \cdot \Delta x,\, i = 1 ..N\right)$

$$\sum_{i=1}^{4} \frac{1}{4} \sin\left(\frac{1}{4}\left(i - \frac{1}{2}\right)\pi\right)\pi \quad (5)$$

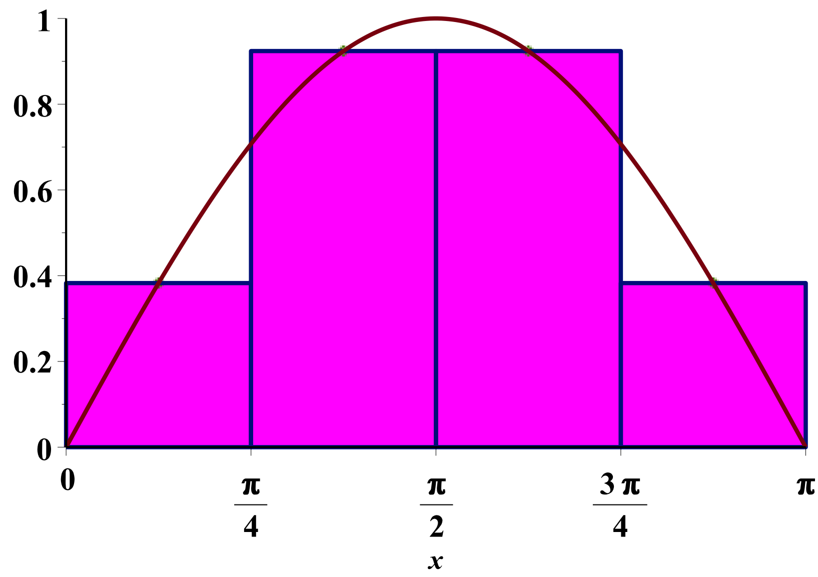
>

> *evalf*(%)

$$2.052344306 \quad (6)$$

>

> *RiemannSum*(*f*(*x*), *x* = *a* ..*b*, *output* = *plot*, *partition* = *N*, *method* = *midpoint*, *boxoptions* = [*filled* = [*color* = *magenta*, *transparency* = .9]], *tickmarks* = ([*spacing*(π/4), *default*]), *thickness* = 2, *font* = [*Roman*, *bold*, 12], *labelfont* = [*Roman*, 10])



A midpoint Riemann sum approximation of $\int_0^{\pi} f(x)\,dx$,

where $f(x) = \sin(x)$ and the partition is uniform. The approximate value of the integral is 2.052344306. Number of subintervals used: 4.

>

> *RiemannSum*(*f* (*x*), *x* = *a* ..*b*, *output* = *value*, *partition* = *N*, *method* = *midpoint*)

$$\frac{1}{4}\sqrt{2-\sqrt{2}}\,\pi + \frac{1}{4}\sqrt{2+\sqrt{2}}\,\pi \qquad \textbf{(7)}$$

> *evalf* (%)

$$2.052344306 \qquad \textbf{(8)}$$

>

Approximate the integral by partitioning $[0, \pi]$ into 20 subintervals of equal width:

> $N := 20$

$$N := 20 \qquad \textbf{(9)}$$

> $\Delta x := \frac{(b-a)}{N}$

(10)

$$\Delta x := \frac{1}{20}\,\pi \qquad \textbf{(10)}$$

> $Sum\left(f\left(a+\left(i-\frac{1}{2}\right)\Delta x\right)\cdot\Delta x,\ i=1..N\right)$

$$\sum_{i=1}^{20} \frac{1}{20}\,\sin\left(\frac{1}{20}\,\left(i-\frac{1}{2}\right)\pi\right)\pi \qquad \textbf{(11)}$$

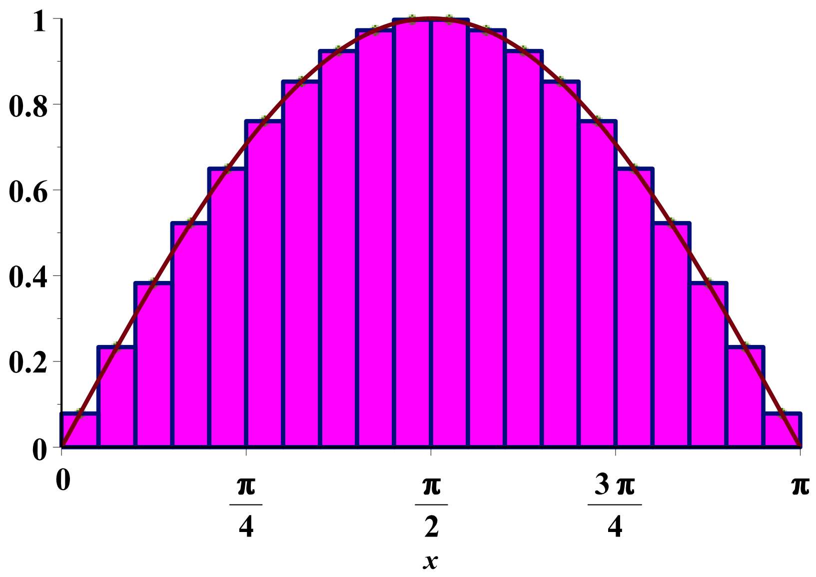
>

> $evalf(\%)$

$$2.002057648 \qquad \textbf{(12)}$$

>

> $RiemannSum(f(x), x=a..b, output=plot, partition=N, method=midpoint, boxoptions=[filled=[color=magenta, transparency=.9]], tickmarks=([spacing(\pi/4), default]), thickness=2, font=[Roman, bold, 12], labelfont=[Roman, 10])$



A midpoint Riemann sum approximation of $\int_0^{\pi} f(x)\,dx$, where $f(x) = \sin(x)$ and the partition is uniform. The approximate value of the integral is 2.002057648. Number of subintervals used: 20.

Precise value of the integral:

> $\int_0^{\pi} \sin(x)\,dx$

$$2 \quad (13)$$

>

Example 2. Use the Midpoint Rule to approximate the integral $\int_0^{3} e^{-x^2}\,dx$ with $N = 10$, 20, and 50 subintervals.

> $g := x \to e^{-x^2}$

$$g := x \to e^{-x^2} \quad (14)$$

>

> $a := 0; b := 3$

$$a := 0$$
$$b := 3 \qquad \textbf{(15)}$$

Approximate the integral by partitioning $[0, 3]$ into 10 subintervals of equal width:

> $N := 10$

$$N := 10 \qquad \textbf{(16)}$$

> $\Delta x := \frac{(b-a)}{N}$

$$\Delta x := \frac{3}{10} \qquad \textbf{(17)}$$

> $Sum\left(g\left(a + \left(i - \frac{1}{2}\right) \cdot \Delta x\right) \cdot \Delta x, i = 1 .. N\right)$

$$\sum_{i=1}^{10} \frac{3}{10} \mathrm{e}^{-\left(\frac{3}{10} i - \frac{3}{20}\right)^2} \qquad \textbf{(18)}$$

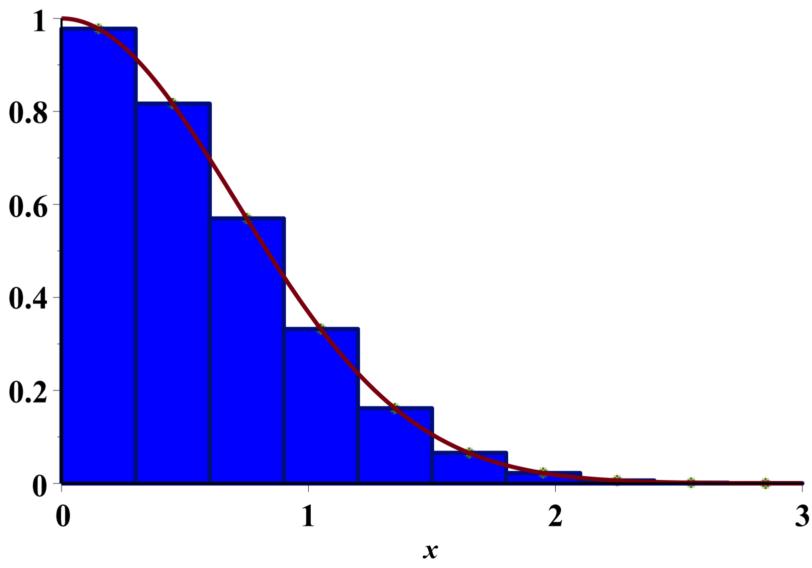
>

> $evalf(\%)$

$$0.8862099170 \qquad \textbf{(19)}$$

>

> *RiemannSum*(*g*(*x*), *x* = *a* ..*b*, *output* = *plot*, *partition* = *N*, *method* = *midpoint*, *boxoptions* = [*filled* = [*color* = *blue*, *transparency* = .9]], *tickmarks* = ([*spacing*(1), *default*]), *thickness* = 2, *font* = [*Roman*, *bold*, 12], *labelfont* = [*Roman*, 10])



A midpoint Riemann sum approximation of $\int_0^3 f(x)\,dx$,

where $f(x) = e^{-x^2}$ and the partition is uniform. The approximate value of the integral is 0.8862099171. Number of subintervals used: 10.

>

> *RiemannSum*$(g(x), x = a..b, output = value, partition = N, method = midpoint)$

$$\frac{3}{10}e^{-\frac{9}{400}} + \frac{3}{10}e^{-\frac{81}{400}} + \frac{3}{10}e^{-\frac{9}{16}} + \frac{3}{10}e^{-\frac{441}{400}} + \frac{3}{10}e^{-\frac{729}{400}} + \frac{3}{10}e^{-\frac{1089}{400}} + \frac{3}{10}e^{-\frac{1521}{400}} + \frac{3}{10}e^{-\frac{81}{16}} + \frac{3}{10}e^{-\frac{2601}{400}} + \frac{3}{10}e^{-\frac{3249}{400}} \quad \textbf{(20)}$$

> *evalf*(%)

$$0.8862099171 \quad \textbf{(21)}$$

>

Approximate the integral by partitioning [0, 3] into 20 subintervals of equal width:

> $N := 20$

$$N := 20 \quad \textbf{(22)}$$

> $\Delta x := \frac{(b-a)}{N}$

$$\Delta x := \frac{3}{20} \quad (23)$$

> $Sum\left(g\left(a+\left(i-\frac{1}{2}\right)\Delta x\right)\cdot \Delta x,\ i=1\,..N\right)$

$$\sum_{i=1}^{20} \frac{3}{20}\, e^{-\left(\frac{3}{20}\, i-\frac{3}{40}\right)^2} \quad (24)$$

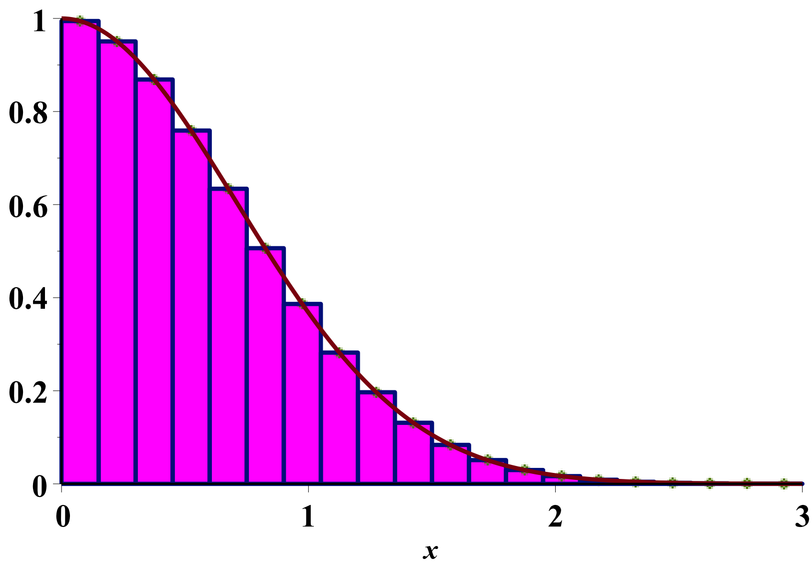
>

> $evalf(\%)$

$$0.8862080289 \quad (25)$$

>

> $RiemannSum(g(x), x=a\,..b, output=plot, partition=N, method=midpoint, boxoptions = [filled=[color=magenta, transparency=.9]], tickmarks=([spacing(1), default]), thickness=2, font=[Roman, bold, 12], labelfont=[Roman, 10])$



A midpoint Riemann sum approximation of $\int_0^3 f(x)\,dx$,

where $f(x) = e^{-x^2}$ and the partition is uniform. The approximate value of the integral is 0.8862080291. Number of subintervals used: 20.

>

Approximate the integral by partitioning [0, 3] into 50 subintervals of equal width:

> $N := 50$

$$N := 50 \tag{26}$$

> $\Delta x := \frac{(b-a)}{N}$

$$\Delta x := \frac{3}{50} \tag{27}$$

> $Sum\left(g\left(a + \left(i - \frac{1}{2}\right)\Delta x\right) \cdot \Delta x, i = 1..N\right)$

$$\sum_{i=1}^{50} \frac{3}{50}\, e^{-\left(\frac{3}{50} i - \frac{3}{100}\right)^2} \tag{28}$$

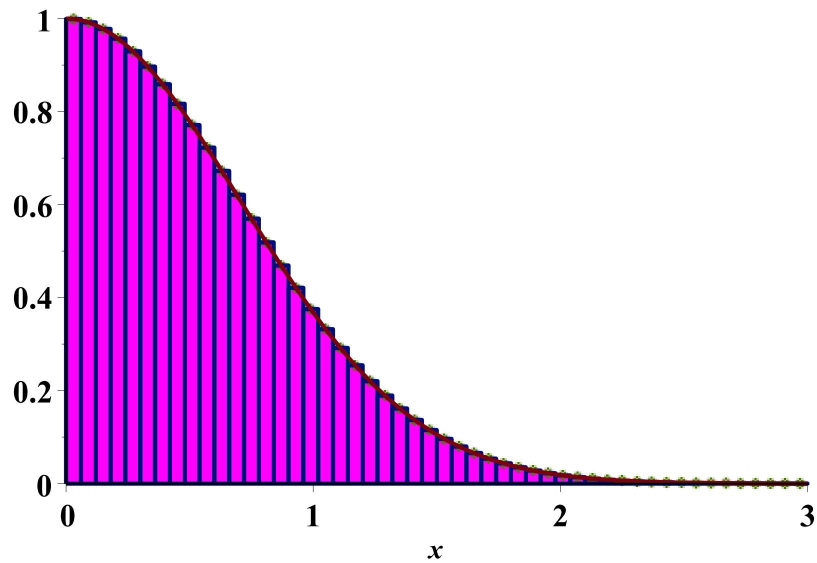
>

> $evalf(\%)$

$$0.8862074590 \qquad (29)$$

>

> $RiemannSum(g(x), x = a..b, output = plot, partition = N, method = midpoint, boxoptions = [filled = [color = magenta, transparency = .9]], tickmarks = ([spacing(1), default]), thickness = 2, font = [Roman, bold, 12], labelfont = [Roman, 10])$



A midpoint Riemann sum approximation of $\int_0^3 f(x)\,dx$, where $f(x) = e^{-x^2}$ and the partition is uniform. The approximate value of the integral is 0.8862074589. Number of subintervals used: 50.

>

Finally, let us compare the above approximations with the value obained using the *error function:*

> $\int_0^3 g(x)\,dx$

$$(30)$$

$$\frac{1}{2}\operatorname{erf}(3)\sqrt{\pi} \qquad \textbf{(30)}$$

> *evalf*(%)

$$0.8862073485 \qquad \textbf{(31)}$$

>

***** END *****