Math 112 - Test 3
November 30, 2018

Name Key

Grade: $\frac{\quad}{80} =$ ____

No calculators. Circle or box-in answers.

1. Use differentiation rules to find the derivative of each of the following functions. Simplify and label each derivative appropriately. Circle or box-in your final answer.

(a) $f(x) = x - 6\cos x + \dfrac{10}{x}$ (6 pts.)

$$f'(x) = \frac{d}{dx}x - 6\frac{d}{dx}\cos x + 10\frac{d}{dx}x^{-1}$$
$$= 1 - 6(-\sin x) + 10(-x^{-2}) = 1 + 6\sin x - \frac{10}{x^2}$$

(b) $y = \dfrac{1 - \ln x}{2x+1}$ (6 pts.)

By the Quotient Rule,

$$y' = \frac{(2x+1)\frac{d}{dx}(1-\ln x) - (1-\ln x)\frac{d}{dx}(2x+1)}{(2x+1)^2}$$
$$= \frac{(2x+1)\left(0-\frac{1}{x}\right) - 2(1-\ln x)}{(2x+1)^2} = \frac{-2-\frac{1}{x}-2+2\ln x}{(2x+1)^2}$$
$$= \frac{2x\ln x - 4x - 1}{x(2x+1)^2}$$

(c) $g(t) = e^{3t} + t^2 \sec t$ (6 pts.)

$$g'(t) = \frac{d}{dt}e^{3t} + \frac{d}{dt}(t^2\sec t)$$
$$= e^{3t}\frac{d}{dt}(3t) + t^2\frac{d}{dt}\sec t + \sec t\frac{d}{dt}t^2$$
$$= 3e^{3t} + t^2\sec t\tan t + 2t\sec t$$

(d) $s = 7\tan(3t) + \dfrac{t}{2} + \sqrt{\pi}.$ (6 pts.)

$$\frac{ds}{dt} = 7\frac{d}{dt}\tan(3t) + \frac{1}{2}\frac{d}{dt}t + \frac{d}{dt}\sqrt{\pi}$$
$$= 7\sec^2(3t)\frac{d}{dt}(3t) + \frac{1}{2}(1) + 0$$ derivative of a constant is 0
$$= 21\sec^2(3t) + \frac{1}{2}$$

/24

(e) $y = \ln(x^2 e^x)$ (6 pts.)

Short way. Since $y = \ln x^2 + \ln e^x = 2\ln x + x$,

$$y' = 2\frac{d}{dx}\ln x + \frac{d}{dx}x = 2\left(\frac{1}{x}\right) + 1 = \frac{2}{x} + 1 = \frac{2+x}{x}.$$

Longer.

$$y' = \frac{1}{x^2e^x}\frac{d}{dx}(x^2e^x) = \frac{1}{x^2e^x}\left(x^2e^x + 2xe^x\right) = \frac{xe^x(x+2)}{x^2e^x} = \frac{x+2}{x}$$

(f) $h(t) = (1+3t^2)^4$ (6 pts.)

$$h'(t) = 4(1+3t^2)^3 \frac{d}{dt}(1+3t^2)$$
$$= 4(1+3t^2)^3(6t)$$
$$= 24t(1+3t^2)^3$$

(g) $y = 3^x + \ln|\sin(3x)|$ (6 pts.)

$$y' = \frac{d}{dx}3^x + \frac{d}{dx}\ln|\sin(3x)|$$
$$= 3^x\ln 3 + \frac{1}{\sin(3x)}\frac{d}{dx}\sin(3x) \quad \text{using } \frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx}$$
$$= 3^x\ln 3 + \frac{3\cos(3x)}{\sin(3x)} = 3^x\ln 3 + 3\cot(3x)$$

2. Let $C(t)$ be the total value of U.S. currency in circulation at time t. The following table gives values of this function from 1980 to 2000, as of December 31, in billions of dollars.

t	1980	1985	1990	1995	2000
$C(t)$	129.8	187.3	271.9	409.3	568.6

(a) Estimate the value of the derivative $C'(1990)$. (4 pts.)

$$C'(1990) \approx \frac{C(1995) - C(1985)}{1995 - 1985} = \frac{409.3 - 187.3}{10}$$
$$= \frac{222 \text{ billion dollars}}{10 \text{ years}} = 22.2 \text{ billion dollars per year}$$

(b) Estimate the value of the derivative $C'(1980)$. (4 pts.)

$$C'(1980) = \lim_{\Delta t \to 0}\frac{C(1980+\Delta t) - C(1980)}{\Delta t} \approx \frac{C(1985) - C(1980)}{1985 - 1980} \quad (\text{taking } \Delta t = 5)$$

So,

$$C'(1980) \approx \frac{187.3 - 129.8}{5} = \frac{57.5}{5} = 11.5 \text{ billion dollars per year}$$

26

3. Find the equation of the tangent line to the curve $y = \sqrt{1 + 4\sin x}$ when $x = 0$. (6 pts.)

when $x = 0$, $y = \sqrt{1+4\sin 0} = \sqrt{1} = 1$.

$y' = \frac{d}{dx}(1+4\sin x)^{1/2} = \frac{1}{2}(1+4\sin x)^{-1/2}\frac{d}{dx}(1+4\sin x) = \frac{2\cos x}{\sqrt{1+4\sin x}}$.

Slope: $m = y'\big|_{x=0} = \frac{2\cos 0}{\sqrt{1+4\sin 0}} = \frac{2(1)}{(1)} = 2$.

Thus, the equation is $y - 1 = 2(x-0)$ or $y = 2x+1$

4. (a) Find $\lim_{x\to\pi} \frac{\sin x}{x}$ (4 pts.)

$$\lim_{x\to\pi} \frac{\sin x}{x} = \frac{\lim_{x\to\pi} \sin x}{\lim_{x\to\pi} x} = \frac{\sin \pi}{\pi} = \frac{0}{\pi} = \boxed{0}$$

(b) Find $\lim_{x\to 0} x\cot(5x)$. (4 pts.)

$$\lim_{x\to 0} x\cot(5x) = \lim_{x\to 0}\left[x\,\frac{\cos(5x)}{\sin(5x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{5}\cdot\frac{5x}{\sin(5x)}\cdot\cos(5x)\right]$$

$$= \frac{1}{5}\lim_{x\to 0}\left[\frac{1}{\frac{\sin(5x)}{5x}}\right]\cdot\lim_{x\to 0}\cos(5x) = \left(\frac{1}{5}\right)\left(\frac{1}{1}\right)(1)$$

$$= \frac{1}{5}$$

5. A table of values for f, g, f', and g' is given in the following table. (6 pts.)

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	3	5	9
3	7	2	7	10

(a) Define the function $h(x) = f(g(x))$. Find $h(2)$.

$h(2) = f(g(2)) = f(3) = \boxed{7}$

(b) Find $h'(2)$.

As $h'(x) = f'(g(x))\,g'(x)$,

$h'(2) = f'(g(2))g'(2) = f'(3)\,g'(2) = (7)(9) = \boxed{63}$

/20

6. The position of a particle, in meters, is given by $s = t^3 - 3t^2 - 5t$, where t is the time in seconds. When does the particle reach a velocity of 139 meters per second? (5 pts.)

The velocity function is

$$v = \frac{ds}{dt} = \frac{d}{dt}(t^3 - 3t^2 - 5t) = 3t^2 - 6t - 5.$$

Now solve $3t^2 - 6t - 5 = 139.$

$$\Rightarrow 3t^2 - 6t - 144 = 0$$

$$\Rightarrow 3(t^2 - 2t - 48) = 0$$

$$\Rightarrow 3(t-8)(t+6) = 0 \Rightarrow t = 8, t = -6$$

Since t denotes time, we conclude the particle reaches a velocity of $139 \frac{m}{s}$ when $t = 8$ seconds

7. (a) Differentiate $y = x^{\sin 2x}$ with respect to x. (4 pts.)

$$\ln y = \ln x^{\sin(2x)} \Rightarrow \ln y = \sin(2x) \cdot \ln x$$

Differentiate both sides wrt x:

$$\frac{d}{dx} \ln y = \frac{d}{dx}\left[\sin(2x) \cdot \ln x\right]$$

$$= \sin(2x) \cdot \frac{1}{x} + 2\cos(2x) \cdot \ln x.$$

$$\Rightarrow \frac{1}{y} y' = \frac{\sin(2x)}{x} + 2(\ln x)\cos(2x)$$

$$\therefore\ y' = x^{\sin(2x)}\left[\frac{\sin(2x)}{x} + 2(\ln x)\cos(2x)\right]$$

(b) Find y' at $x = 1$. (Simplify the answer as much as is possible.) (1 pt.)

When $x = 1$,

$$y' = 1\left[\sin 2 + 2(\ln 1)\cos 2\right] = \sin 2 \quad \text{since } \ln 1 = 0.$$

Extra Credit. (4 pts.)
Find the critical numbers of the function $f(x) = x^3 + 3x^2 + 3$ on the interval $[-1, 10]$.

$$f'(x) = 3x^2 + 6x$$

$$f'(x) = 0 \Rightarrow 3x(x+2) = 0 \Rightarrow x = 0, x = -2$$

$\boxed{\text{The critical number in } [-1,10] \text{ is } x = 0}$

14