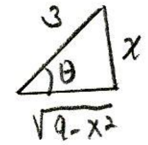
Quiz 17

4p491 Evaluate $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$.

Soln. Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$.

Reference triangle: 

$\frac{\sqrt{9-x^2}}{3} = \cos\theta$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta$$

$$= \frac{9}{2}\int [1-\cos 2\theta]\,d\theta = \frac{9}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right] + C$$

$$= \frac{9}{2}\theta - \frac{9}{4}(2\sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

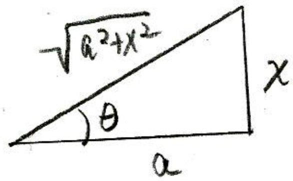
$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

$$= \frac{1}{2}\left[9\sin^{-1}\left(\frac{x}{3}\right) - x\sqrt{9-x^2}\right] + C$$

$\boxed{7\ p491}$ Evaluate $\displaystyle\int_0^a \frac{dx}{(a^2+x^2)^{3/2}}$, where $a>0$.

Soln. Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$

When $x=0$, $\tan\theta = 0 \Rightarrow \theta = 0$. When $x = a$, $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$



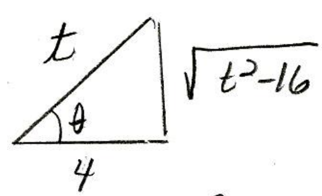
$$\Rightarrow \frac{\sqrt{a^2+x^2}}{a} = \sec\theta$$

$$\int_0^a \frac{dx}{(a^2+x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{(a\sec\theta)^3} = \frac{1}{a^2}\int_0^{\pi/4} \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{a^2}\int_0^{\pi/4} \cos\theta\, d\theta = \frac{1}{a^2}\Big[\sin\theta\Big]_0^{\pi/4} = \frac{1}{a^2}\Big[\sin\frac{\pi}{4} - \sin 0\Big]$$

$$= \frac{\sqrt{2}}{2a^2}$$

$\boxed{8\ p491}$ Evaluate $\displaystyle\int \frac{dt}{t^2\sqrt{t^2-16}}$.



Soln.
Let $t = 4\sec\theta$. $dt = 4\sec\theta\tan\theta\, d\theta$

$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \int \frac{4\sec\theta\tan\theta\, d\theta}{(16\sec^2\theta)\, 4\tan\theta} = \frac{1}{16}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{16}\int \cos\theta\, d\theta = \frac{1}{16}\sin\theta + C$$

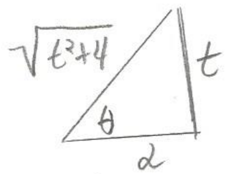
$$= \frac{1}{16}\,\frac{\sqrt{t^2-16}}{t} + C$$

$$= \frac{\sqrt{t^2-16}}{16t} + C$$

12 p491 Evaluate $\int_0^2 \frac{dt}{\sqrt{4+t^2}}$.

Soln. Let $t = 2\tan\theta$. Then $dt = 2\sec^2\theta \, d\theta$

$t=0 \Rightarrow \theta = 0$. And $t=2 \Rightarrow \theta = \frac{\pi}{4}$.



$\Rightarrow \frac{\sqrt{t^2+4}}{2} = \sec\theta$.

$$\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \int_0^{\pi/4} \frac{2\sec^2\theta \, d\theta}{2\sec\theta} = \int_0^{\pi/4} \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| \Big|_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln|\sqrt{2}+1| - \ln|1+0|$$

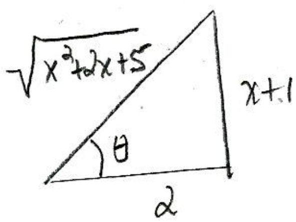
$$= \ln(1+\sqrt{2})$$

$\boxed{23\ p491}$ Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$.

Soln. Completing the square, we have: $x^2+2x+5 = x^2+2x+1-1+5$
$= (x+1)^2+4$.

So, $$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x+1)^2+4}} =: I$$

Let $x+1 = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$.



$\Rightarrow \sqrt{x^2+2x+5} = 2\sec\theta$

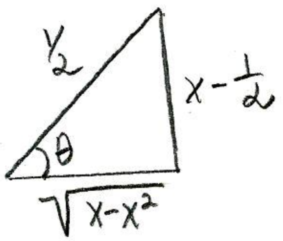
So,
$$I = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left| \frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2} \right| + C$$
$$= \ln\left(\frac{\left|\sqrt{x^2+2x+5}+(x+1)\right|}{2}\right) + C$$
$$= \ln\left| x+1+\sqrt{x^2+2x+5} \right| + K \qquad (K = C - \ln 2)$$

24 p 491 Evaluate $\int_0^1 \sqrt{x-x^2}$.

Soln. Complete the square.

$$x - x^2 = -(x^2 - x) = -\left(x^2 - x + \frac{1}{4}\right) + \frac{1}{4} = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2$$

So this is a $\sqrt{a^2-u^2}$ type problem, where $a = \frac{1}{2}$ and $u = x - \frac{1}{2}$.



Let $x - \frac{1}{2} = \frac{1}{2}\sin\theta$; $\sin\theta = \dfrac{x-\frac{1}{2}}{\frac{1}{2}}$

$$dx = \frac{1}{2}\cos\theta \, d\theta$$

$$\sqrt{x-x^2} = \frac{1}{2}\cos\theta$$

When $x=0$, $\sin\theta = -1 \Rightarrow \theta = -\pi/2$.

When $x=1$, $\sin\theta = 1 \Rightarrow \theta = \pi/2$.

$$\int_0^1 \sqrt{x-x^2}\,dx = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\cos\theta\right)\left(\frac{1}{2}\cos\theta\right) d\theta = \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\cdot 2\int_0^{\pi/2} \cos^2\theta \, d\theta \quad \text{(as } \cos\theta \text{ is an even fnct)}$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right]$$

$$= \frac{\pi}{8}$$