Math 122 - Test 2
March 25, 2019

Name Key

Grade: $\frac{\quad}{80} = \_\_\_\_$

Show your work. Points will be deducted for inadequate explanations, using notation incorrectly, and not making obvious and conventional simplifications.

1. Evaluate using the methods of substitution and/or integration by parts. (40 pts.)

(a) $\int_{-1}^{0} \frac{6}{2-3x}\,dx$

Let $u = 2-3x$. Then $du = -3\,dx$. So $dx = -\frac{1}{3}\,du$.

$x=-1 \Rightarrow u = 2-3(-1) = 5$; $x = 0 \Rightarrow u = 2$.

$$\int_{-1}^{0} \frac{6\,dx}{2-3x} = \int_{5}^{2} \frac{6}{u}\left(-\frac{1}{3}\,du\right) = -2\int_{5}^{2} \frac{du}{u}$$

$$= 2\int_{2}^{5} \frac{du}{u} = 2\Big[\ln u\Big]_{2}^{5}$$

$$= 2(\ln 5 - \ln 2)$$

$$= 2\ln\left(\frac{5}{2}\right).$$

(b) $\int \tan(\theta/4)\,d\theta$

$$\int \tan\left(\frac{\theta}{4}\right) d\theta = \int \frac{\sin\left(\frac{\theta}{4}\right)}{\cos\left(\frac{\theta}{4}\right)}\,d\theta$$

(Let $u = \cos\left(\frac{\theta}{4}\right)$. Then $du = -\frac{1}{4}\sin\left(\frac{\theta}{4}\right)d\theta$ $\Rightarrow \sin\left(\frac{\theta}{4}\right)d\theta = -4\,du$)

$$= -4\int \frac{du}{u}$$

$$= -4\ln|u| + C$$

$$= -4\ln\left|\cos\left(\frac{\theta}{4}\right)\right| + C$$

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(c) $\int x^3 \cos\left(\frac{x}{2}\right) dx$

LIATE

$u = x^3 \quad dv = \cos\left(\frac{x}{2}\right) dx$

	$\frac{d}{dx}[\cdot]$	$\int [\cdot] dx$
(+)	$u = x^3$	$\cos\left(\frac{x}{2}\right) dx$
(−)	$3x^2$	$2\sin\left(\frac{x}{2}\right)$
(+)	$6x$	$-4\cos\left(\frac{x}{2}\right)$
(−)	6	$-8\sin\left(\frac{x}{2}\right)$
	0	$16\cos\left(\frac{x}{2}\right)$

$$\int x^3 \cos\left(\frac{x}{2}\right) dx = 2x^3 \sin\left(\frac{x}{2}\right) + 12x^2 \cos\left(\frac{x}{2}\right) - 48x \sin\left(\frac{x}{2}\right) - 96\cos\left(\frac{x}{2}\right) + C$$

(d) $\int \frac{\cos x}{\sqrt{1-\sin^2 x}} dx \quad$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Let $u = \sin x$. Then $du = \cos x \, dx$.

$$\int \frac{\cos x}{\sqrt{1-\sin^2 x}} dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\sin x) + C$$

Since $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $\sin^{-1}(\sin x) = x$. Thus,

$$\int \frac{\cos x}{\sqrt{1-\sin^2 x}} dx = x + C.$$

Or: As $\sin^2 x + \cos^2 x = 1$, $\sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = \sqrt{(\cos x)^2} = |\cos x|$.

For $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $\cos x \geq 0$. Thus, $|\cos x| = \cos x$.

Hence, $\int \frac{\cos x \, dx}{\sqrt{1-\sin^2 x}} = \int \frac{\cos x}{\cos x} dx = \int dx = x + C$.

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(e) $\int \cos\sqrt{x}\,dx$

Let $u = x^{1/2}$. Then $du = \frac{1}{2}x^{-1/2}dx \Rightarrow \frac{dx}{\sqrt{x}} = 2\,du \Rightarrow dx = 2u\,du.$

$$\int \cos\sqrt{x}\,dx = 2\int u\cos u\,du = 2\int t\cos t\,dt$$

LIATE

$u = t$ ⟵ ⟶ $dv = \cos t\,dt$

$u = t \quad (+) \quad dv = \cos t\,dt$

$du = dt \quad (-) \quad v = \sin t$

$$\int t\cos t\,dt = t\sin t - \int \sin t\,dt = t\sin t + \cos t + K$$

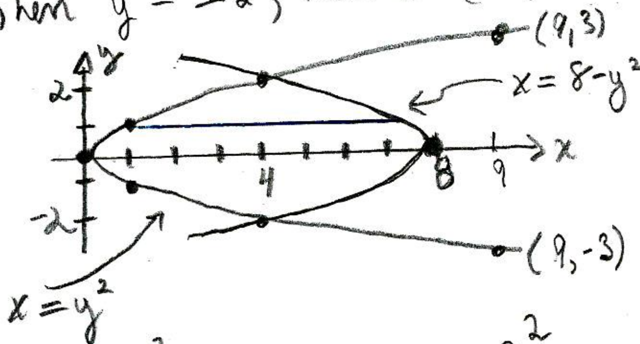
$$\therefore \int \cos\sqrt{x}\,dx = 2\left[t\sin t + \cos t\right] + 2K$$

$$= 2\left[\sqrt{x}\sin(\sqrt{x}) + \cos(\sqrt{x})\right] + C$$

2. Find the area of the region bounded by the curves $x = y^2$ and $x = 8 - y^2$. (8 pts.)

$y^2 = 8 - y^2 \Rightarrow 2y^2 = 8 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2.$

when $y = \pm 2$, then $x = (\pm 2)^2 = 4.$



$$A = \int_{-2}^{2}(x_R - x_L)\,dy = \int_{-2}^{2}\left[(8-y^2) - y^2\right]dy = \int_{-2}^{2}(8-2y^2)\,dy$$

$$= 2\int_{-2}^{2}(4-y^2)\,dy = 4\int_{0}^{2}(4-y^2)\,dy \qquad (\text{as } g(y) := 4-y^2 \text{ is an even fnct})$$

$$= 4\left[4y - \frac{y^3}{3}\right]_0^2 = 4\left[8 - \frac{8}{3} - 0\right] = 4\left(\frac{24}{3} - \frac{8}{3}\right) = \frac{64}{3}\text{ units}^2$$

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3. Differentiate the function (8 pts.)

$$f(x) = \int_{e^x}^{5-x} \sin t^2 \, dt.$$

Using some of the integral properties,

$$f(x) = \int_{e^x}^{0} \sin t^2 \, dt + \int_{0}^{5-x} \sin t^2 \, dt = -\int_{0}^{e^x} \sin t^2 \, dt + \int_{0}^{5-x} \sin t^2 \, dt.$$

Differentiating, we have by the FTC-1 and the Chain Rule

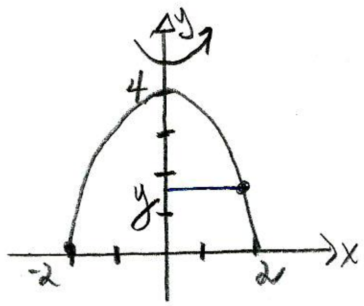
$$f'(x) = -\frac{d}{dx}\int_{0}^{e^x} \sin(t^2)\, dt + \frac{d}{dx}\int_{0}^{5-x} \sin t^2 \, dt$$

$$= -\sin((e^x)^2)\frac{d}{dx}e^x + \sin((5-x)^2)\frac{d}{dx}(5-x)$$

$$= -e^x \sin(e^{2x}) - \sin(5-x)^2$$

4. Compute the volume of the solid obtained by rotating the region bounded by $y = 4 - x^2$ and the x-axis about the **y-axis.** (8 pts.)

$y = 4 - x^2$ is a parabola with vertex at $(0,4)$ and x-intercepts ± 2.



Cross-sectional area:

$$A(y) = \pi x^2 = \pi(4-y).$$

Volume:

$$V = \int_0^4 A(y)\, dy = \int_0^4 \pi(4-y)\, dy = \pi\left[4y - \frac{1}{2}y^2\right]_0^4$$

$$= \pi\left[16 - 8\right] = 8\pi \text{ units}^3$$

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5. Let $C(x)$ denote the cost (in dollars) for a small family business to produce x units of a certain commodity. If the marginal cost is $C'(x) = 0.04x + 2$, what is the increase in cost when the number of units produced is increased from 10 to 20? (8 pts.)

By the FTC-2, $\int_{10}^{20} C'(x)\,dx = C(x)\Big|_{10}^{20} = C(20) - C(10) = \Delta C$

So, $\Delta C = \int_{10}^{20} (.04x+2)\,dx = \left[.02x^2+2x\right]_{10}^{20}$

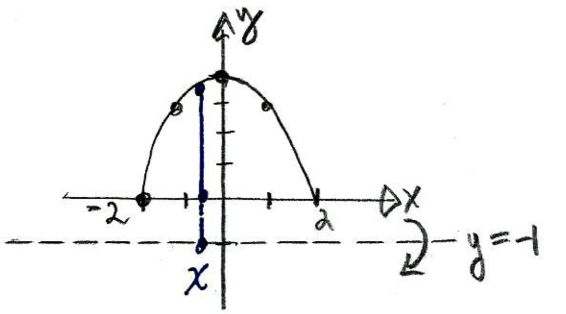
$= .02(20^2-10^2) + 2(20-10)$

$= .02(300) + 2(10)$

$= 6.00 + 20 = 26.$

$\Delta C = 26$ dollars

7. Set up the integral for the volume of the solid obtained by rotating the region bounded by the curve $y = 4 - x^2$ and the x-axis about the line $y = -1$. (Do not evaluate the integral.) (8 pts.)



For a given $x \in [-2,2]$,

$A(x) = \pi r_o^2 - \pi r_i^2$

where $r_o = 1 + (4-x^2) = 5-x^2$ and

$r_i = 1.$

So, $A(x) = \pi\left[(5-x^2)^2 - 1^2\right]$

$= \pi\left[25 - 10x^2 + x^4 - 1\right] = \pi\left[x^4 - 10x^2 + 24\right]$

$V = \int_{-2}^{2} A(x)\,dx = \pi \int_{-2}^{2} \left[x^4 - 10x^2 + 24\right] dx$

or $V = 2\pi \int_0^2 (x^4 - 10x^2 + 24)\,dx$

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Extra Credit. Evaluate the definite integral (4 pts.)

$$\int_{-2}^{2} x^5 e^{x^2}\, dx.$$

Short way:

The integrand is

$$f(x) := x^5 e^{x^2}.$$

Note that

$$f(-x) = (-x)^5 e^{(-x)^2} = -x^5 e^{x^2} = -f(x).$$

So f is an odd function. Thus,

$$\boxed{\int_{-2}^{2} x^3 e^{x^2}\, dx = 0}$$

Long way: $\int x^5 e^{x^2} dx = \int x^4 e^{x^2}\, x\, dx = \frac{1}{2}\int (x^2)^2 e^{x^2}\, 2x\, dx$

Let $t = x^2$. Then $dt = 2x\,dx$. So

$$\int x^5 e^{x^2} dx = \frac{1}{2}\int t^2 e^t\, dt$$

	$\frac{d}{dt}[\cdot]$	$\int [\cdot]\,dx$
(+)	$u = t^2$	$e^t\, dt = dv$
(−)	$2t$	e^t
(+)	2	e^t
(−)	0	e^t

$$\int t^2 e^t\, dt = e^t(t^2 - 2t + 2) + C$$

$$\therefore \int x^5 e^{x^2} dx = \frac{1}{2} e^{x^2}(x^4 - 2x^2 + 2) + C$$

Therefore

$$\int_{-2}^{2} x^5 e^{x^2} dx = \frac{1}{2} e^{x^2}(x^4 - 2x^2 + 2)\Big|_{-2}^{2} = \boxed{0}$$