# Quiz 4

**P1 (#4 p. 249)** The length of a rectangle is increasing at a rate of $8\ \frac{cm}{s}$ and its width is increasing at a rate of $3\ \frac{cm}{s}$. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

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<u>Solution</u> Let $x$ & $y$ denote the lengths of the sides in centimeters as shown here:


<u>Given:</u> $\frac{dx}{dt} = 3\ \frac{cm}{s}$ , $\frac{dy}{dt} = 8\ \frac{cm}{s}$ where $t$ denotes the time in seconds.

Let $A$ denote the area of the rectangle.

<u>Objective.</u> To find $\frac{dA}{dt}$ when $x = 10\,cm$ & $y = 20\,cm$.

Since $A = xy$,

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = 8x + 3y.$$

When $x = 10$ & $y = 20$,

$$\frac{dA}{dt} = 10(8) + 3(20) = 80 + 60 = 140$$

$$\therefore \quad \frac{dA}{dt} = 140\ \frac{cm^2}{s}$$

$\boxed{\text{P2 (\#6 p. 249)}}$ The radius of a sphere is increasing at a rate of 4 mm/s. How fast is the volume increasing when the diameter is 80 mm?

Soln. Let $t$ denote the time in seconds. Let $r$ denote the radius of the sphere in millimeters. The volume $V$ of a sphere of radius $r$ is

$$V = \frac{4}{3}\pi r^3 \qquad (\text{Cf. Reference p. 1}).$$

Given. $\frac{dr}{dt} = 4 \frac{mm}{s}$

Objective. To find $\frac{dV}{dt}$ when $r = 40mm$. ($r$ = diameter/2)

Differentiating wrt $t$, we get

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2 \frac{dr}{dt}\right) = 4\pi r^2 \frac{dr}{dt}.$$

When $r = 40mm$,

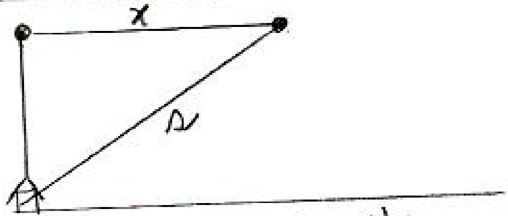
$$\frac{dV}{dt} = 4\pi (40mm)^2 \, 4\frac{mm}{s} = 25{,}600\pi \frac{mm^3}{s}$$

$$= 80{,}424.8 \frac{mm^3}{s}.$$

(to the nearest tenth)

P3 (#13 p249) A plane flying horizontally at an altitude of 1 mile and a speed of 500 $\frac{mi}{h}$ passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles from the station.

Solution.




Let $x$ be the horizontal distance in miles between the plane & radar station. Let $t$ be the time in hours. Let $s$ be the distance in miles between the plane & radar station (see figure).

Given. $\frac{dx}{dt} = 500$ mi/h.

Objective. To find $\frac{ds}{dt}$ when $s = 2$.

By the Pythagorean theorem, we get the equation relating the variables: $s^2 = 1^2 + x^2$. $\Rightarrow$ $s^2 = 1 + x^2$

Differentiating wrt $t$, we obtain the eqn relating the rates:

$$\frac{d}{dt} s^2 = \frac{d}{dt}(1+x^2) \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \frac{ds}{dt} = \left(\frac{x}{s}\right)\frac{dx}{dt}$$

when $s = 2$, then $x = \sqrt{s^2-1} = \sqrt{2^2-1} = \sqrt{3}$.

Thus, $\frac{ds}{dt} = \frac{\sqrt{3}}{2}(500) = 250\sqrt{3}$ mi/h $\approx 433$ mph

Problem 4 (#16 p. 249) At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 $\frac{km}{h}$ and ship B is sailing north at 25 $\frac{km}{h}$. How fast is the distance between the ships changing at 4:00 PM?

(a) What quantities are given?

Let $t$ = the time in hours after 12:00 PM.

Let $x$ = distance in km between ship A and the initial position of ship B at $t=0$

Let $y$ = distance in km between ship B and its position at $t=0$.

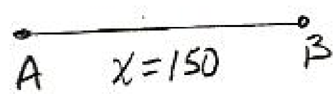
Let $s$ denote the distance in km between the ships

Given. At $t=0$, $x=150$ and $y=0$.

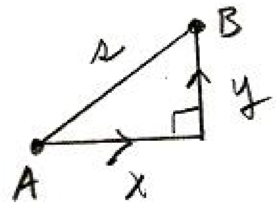
For $0 \le t \le 4$, $\frac{dx}{dt} = -35 \frac{km}{h}$ and $\frac{dy}{dt} = 25 \frac{km}{h}$.

(b) The unknown is $\frac{ds}{dt}$.

(c) At noon ($t=0$)




For $0 < t < 4$




(d) The eqn relating the quantities is $s^2 = x^2 + y^2$.

(e) Taking the derivative wrt the time $t$, we obtain

$$\frac{d}{dt} s^2 = \frac{d}{dt}(x^2+y^2) \Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \boxed{s \frac{ds}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}}$$

At 4:00 PM, $t=4$. And $x(4) = 150 - 35(4) = 10$, $y(4) = 25(4) = 100$. Also, $s = \sqrt{10^2 + 100^2} = \sqrt{100(1+100)} = 10\sqrt{101}$.

$$\therefore 10\sqrt{101} \frac{ds}{dt} = 10(-35) + 100(25) = 2150 \Rightarrow \frac{ds}{dt} = \frac{2150}{10\sqrt{101}}$$

$$\Rightarrow \boxed{\frac{ds}{dt} = \frac{215}{\sqrt{101}} \text{ mph} = 21.39 \text{ mph}}$$