Quiz 3

P1 (#30, p. 215)

Use implicit differentiation to find an equation of the tangent line to $x^{2/3} + y^{2/3} = 4$ at $(-3\sqrt{3}, 1)$.

<u>Solution.</u>

$$\frac{d}{dx}\left(x^{2/3} + y^{2/3}\right) = \frac{d}{dx}4 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \text{ where } y' = \frac{dy}{dx}.$$

Hence, we have $y^{-1/3}y' = -x^{-1/3}$

or $$y' = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} \quad \text{if } x \neq 0$$

when $x = -3\sqrt{3}$ and $y = 1$, then

$$y' = -\frac{\sqrt[3]{1}}{(-3\sqrt{3})^{1/3}} = -\frac{1}{(-3^{3/2})^{1/3}} = -\frac{1}{-3^{1/2}} = \frac{1}{\sqrt{3}},$$

which is the slope of the tangent line to the curve at $(-3\sqrt{3}, 1)$. And so its equation is

$$y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}x + 3 + 1$$

$$\Rightarrow \quad \boxed{y = \frac{\sqrt{3}}{3}x + 4}$$

P2 #40 p. 216 If $x^2+xy+y^3=1$, find the value of y'' at the point where $x=1$.

<u>Solution.</u> First let's find the y-coordinate of the point when $x=1$:

$$1^2+(1)y+y^3=1 \Rightarrow y+y^3=0 \Rightarrow y(1+y^2)=0 \Rightarrow y=0.$$

So the point is $(1,0)$.

Now let's find y'. Differentiating implicitly, we get

$$\frac{d}{dx}\left[x^2+xy+y^3\right]=\frac{d}{dx}1 \Rightarrow 2x+xy'+y+3y^2y'=0.$$

Solving for y', we get

$$y'(x+3y^2)=-2x-y \Rightarrow y'=-\frac{2x+y}{x+3y^2}$$

At the point $(1,0)$,

$$y'=-\frac{2(1)+0}{1+3(0)^2}=\underline{\underline{-2}}.$$

In order to find y'', let's differentiate the encircled expression. By the quotient rule,

$$y''=-\frac{d}{dx}\left[\frac{2x+y}{x+3y^2}\right]=-\frac{(x+3y^2)\frac{d}{dx}(2x+y)-(2x+y)\frac{d}{dx}(x+3y^2)}{(x+3y^2)^2}$$

$$=-\frac{(x+3y^2)(2+y')-(2x+y)(1+6yy')}{(x+3y^2)^2}.$$

Letting $x=1$, $y=0$, $y'=-2$, we have

$$y''=-\frac{(1)(0)-(2)(1)}{1}=2.$$

$$\boxed{\text{Thus, } y''=2 \text{ when } x=1.}$$

P3 (#53, p216)

Differentiate $F(x) = x\sec^{-1}(x^3)$.

Soln. $F'(x) = \frac{d}{dx} x\sec^{-1}(x^3) = x\frac{d}{dx}\sec^{-1}(x^3) + \sec^{-1}(x^3)\frac{d}{dx}x$

$$= x\frac{1}{x^3\sqrt{(x^3)^2-1}} \cdot \frac{d}{dx}x^3 + \sec^{-1}(x^3)\cdot 1$$

$$= \frac{3x^3}{x^3\sqrt{x^6-1}} + \sec^{-1}(x^3)$$

$$\therefore\ F'(x) = \frac{3}{\sqrt{x^6-1}} + \sec^{-1}(x^3)$$

P4 (#58 p216) Differentiate $y = \cos^{-1}(\sin^{-1}t)$.

Soln. $y' = \frac{d}{dt}\cos^{-1}(\sin^{-1}t) = -\frac{1}{\sqrt{1-(\sin^{-1}t)^2}} \cdot \frac{d}{dt}\sin^{-1}t$

$$= -\frac{1}{\sqrt{1-(\sin^{-1}t)^2}} \cdot \frac{1}{\sqrt{1-t^2}}$$

$$= -\frac{1}{\sqrt{1-t^2}\cdot\sqrt{1-(\sin^{-1}t)^2}}$$