Quiz 12

P1 (#15 p.124) Show that $f(x) = x + \sqrt{x-4}$ is continuous on $[4,\infty)$.

Soln. For an arbitrary $a > 4$, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(x + \sqrt{x-4}\right) = \lim_{x\to a} x + \lim_{x\to a}\sqrt{x-4}$$ Sum Law (Law 1)

$$= a + \sqrt{\lim_{x\to a}(x-4)}$$ Trivial Law (Law 8); Root Law (L11) Note. $a-4>0$

$$= a + \sqrt{a-4}.$$ Direct Substitution (bottom p.97)

Since $f(a) = a + \sqrt{a-4}$, we see that $\lim_{x\to a} f(x) = f(a)$. This shows f is continuous on the open interval (a,∞).

Also, we see that

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\left(x + \sqrt{x-4}\right) = \lim_{x\to 4^+} x + \lim_{x\to 4^+}\sqrt{x-4}$$

$$= 4 + \sqrt{0} = 4.$$

Since $f(4) = 4 + \sqrt{4-4} = 4 + \sqrt{0} = 4$, we have

$$\lim_{x\to 4^+} f(x) = f(4).$$

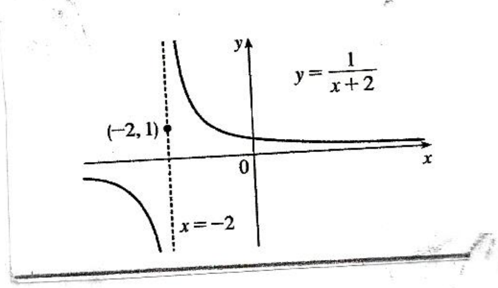
This says that f is continuous from the right at $x=4$.

We conclude that f is continuous on $[4,\infty)$

P2 (#18 p.124) Explain why

$$f(x) = \begin{cases} \dfrac{1}{x+2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases}$$

is discontinuous at $x=-2$.



Soln. We see that $f(-2) = 1$. However,

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-}\frac{1}{x+2} = -\infty \text{ while } \lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}\frac{1}{x+2} = \infty.$$

Since $\lim_{x\to -2} f(x)$ does not exist, f is discontinuous at $x=-2$.

P3 (#31, p.149)

Find $f'(a)$ for $f(x) = 3x^2 - 4x + 1$.

Soln.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{3(a+h)^2 - 4(a+h) + 1 - (3a^2 - 4a + 1)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3a^2} + 6ah + 3h^2 - \cancel{4a} - 4h + \cancel{1} - \cancel{3a^2} + \cancel{4a} - \cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{6ah + 3h^2 - 4h}{h} = \lim_{h\to 0} \frac{h(6a+3h-4)}{h}$$

$$= \lim_{h\to 0} (6a + 3h - 4) = 6a - 4$$

$$\boxed{\therefore\ f'(a) = 6a - 4}$$

P4 (#36 p149) Find $f'(a)$ for $f(x) = \frac{4}{\sqrt{1-x}}$.

Soln.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\frac{4}{\sqrt{1-(a+h)}} - \frac{4}{\sqrt{1-a}}}{h}$$

$$= \lim_{h\to 0} \frac{4}{h}\left[\frac{1}{\sqrt{1-a-h}} - \frac{1}{\sqrt{1-a}}\right]$$

$$= 4 \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{1-a} - \sqrt{1-a-h}}{\sqrt{1-a-h}\cdot\sqrt{1-a}}\right]$$

$$= 4 \lim_{h\to 0} \frac{1}{h}\left[\frac{\sqrt{1-a} - \sqrt{1-a-h}}{\sqrt{1-a-h}\cdot\sqrt{1-a}} \cdot \frac{\sqrt{1-a} + \sqrt{1-a-h}}{\sqrt{1-a} + \sqrt{1-a-h}}\right]$$

$$= 4 \lim_{h\to 0} \left[\frac{1-a-(1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}\left(\sqrt{1-a} + \sqrt{1-a-h}\right)}\right]$$

$$= 4 \lim_{h\to 0} \frac{h}{h\sqrt{1-a-h}\,\sqrt{1-a}\left(\sqrt{1-a} + \sqrt{1-a-h}\right)}$$

$$= 4 \lim_{h\to 0} \frac{1}{\sqrt{1-a-h}\,\sqrt{1-a}\left(\sqrt{1-a} + \sqrt{1-a-h}\right)}$$

$$= 4 \frac{1}{\sqrt{1-a}\sqrt{1-a}\left(\sqrt{1-a}+\sqrt{1-a}\right)}$$

$$= 4 \frac{1}{(1-a)\left(2\sqrt{1-a}\right)}$$

$$\therefore \quad f'(a) = \frac{2}{(1-a)\sqrt{1-a}} = \frac{2}{(1-a)^{3/2}}$$

$\boxed{\text{P5 (\#38 p149)}}$

The limit $\lim_{h\to 0} \frac{e^{-2+h}-e^{-2}}{h}$ represents the derivative of some function f at some number a. State f and a.

<u>Soln.</u> If $f(x) = e^x$ and $a = -2$, then we have

$$f'(2) = \lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h\to 0} \frac{e^{-2+h}-e^{-2}}{h}.$$

<u>Conclusion.</u> $f(x) = e^x$ and $a = -2$.